Rob Bayer

Math 1B Worksheet

July 20, 2010

Work through the following practice problems as a group. As always with practice tests, the inclusion or absence of certain types of problems should not be taken as an indication of what will or will not be on the actual midterm. And don't freak out: this is longer than the actual midterm will be.

1. Quick Answers
   (a) Identify each of the following as (L) linear, (S) separable, (B) both, or (N) neither.
      i. $y' = xy$
      ii. $xy' = y(y+1)\cos x$
      iii. $yy' = xy$
      iv. $yy' = x + y$

   (b) Sketch a few solution curves to $y' = y(y-1)(y+2)$, making sure to clearly indicate all equilibrium solutions.

   (c) Write the form of a particular solution to each of the following. Be sure to multiply by x where necessary.
      i. $y'' + y = e^x$
      ii. $y'' + y = \cos x$
      iii. $y'' + y = e^x \cos x + x^2 e^x \sin x$
      iv. $y'' + 2y' + y = e^{-x}$ (be careful!)
      v. $y'' + 2y' + y = \sin x + 5 + x + e^x$

2. Consider the differential equation $y'' + 4y = 0$
   (a) Find the general solution
   (b) For each of the following boundary conditions, determine whether there is one solution, no solutions, or infinitely many solutions to the associated boundary value problem. For one or infinitely many, find the solution
      i. $y(0) = 0, y(\pi) = 1$
      ii. $y(0) = 0, y(\pi) = 0$
      iii. $y(0) = 0, y(\pi/4) = 1$

3. Find the solution to each of the following
   (a) $y' + 2xy = y$
   (b) $y' - 2xy = 3x^2 e^{x^2}$; $y(0) = 3$
   (c) $y'' - 2y' + y = \cos x + 3\sin x$
   (d) $x^2 y' = x^2 + y^2 + yx$ (Hint: $v = y/x$)
   (e) $y' - x = x \sin y$
   (f) $xy' + x = e^{y/x} + y$ (Hint: $v = y/x$)
   (g) $y' + y = xy^3$ (Hint: $v = y^{-2}$)
   (h) $y'' + y = \csc x$; $y(0) = 1, y(\pi/2) = 2$
   (i) $y'' + y = x \sin x$
   (j) $y'' + 2y' + 2y = e^x + x + e^{-x}\sec^2 x$

4. A 100L tank initially contains 10kg of salt. A bag of salt is poured into the tank at a rate of 1kg/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 2L/min.
   (a) Setup an initial value problem whose solution will give the amount of salt in the tank at time t
   (b) Solve your differential equation from part (a)

5. A certain population of rabbits grows at a rate proportional to the product of its current population and the log of its current population. If there are currently 4 rabbits, and 1 month later there are 7, how many rabbits will there be in 1 year? In t months?

1. Quick Answers

i) B   ii) S   iii) B   iv) N

2. $y' = y(y-1)(y+2) \longrightarrow \int \frac{dy}{y(y-1)(y+2)}^{①} = \int dx$

① $\frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+2} = 1$

$A(y-1)(y+2) + By(y+2) + Cy(y-1) = 1$

$y=1)$ $\quad 3B = 1 \longrightarrow B = 1/3$

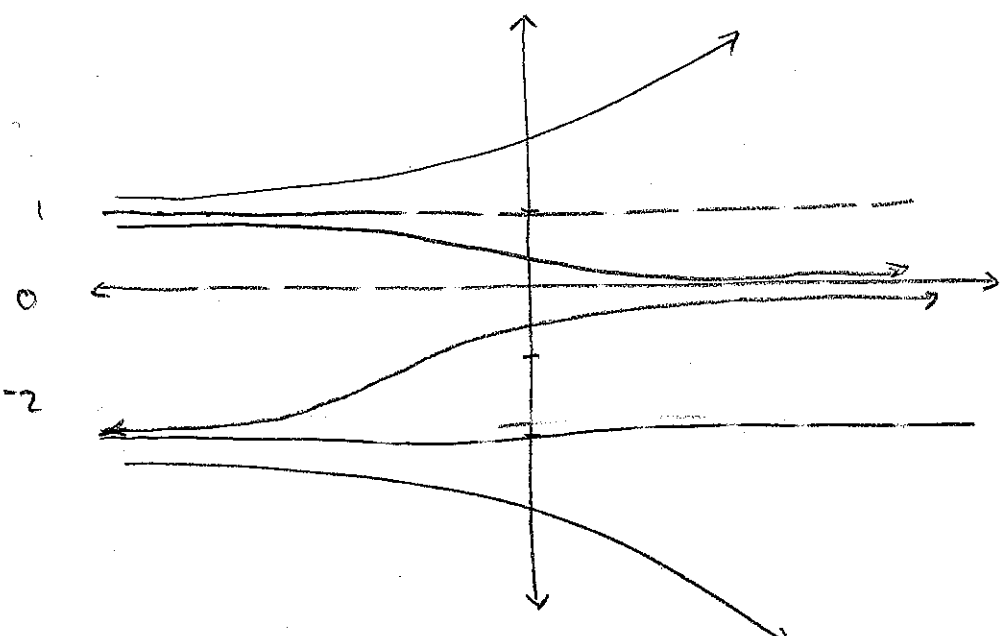
$y=-2)$ $\quad 6C = 1 \longrightarrow C = 1/6$

$y=0)$ $\quad -2A = 1 \quad A = -1/2$

$\ominus\frac{1}{2}\int \frac{1}{y} + \frac{1}{3}\int \frac{1}{y-1} + \frac{1}{6}\int \frac{1}{y+2}$

$\hookrightarrow \frac{1}{2}\ln\left|\frac{1}{y}\right| + \frac{1}{3}\ln|y-1| + \frac{1}{6}\ln|y+2| + C = X$

at $y = 0, 1, -2$, $y' = 0$, so the



MATH 1B WORKSHEET

1c

(i) $y'' + y = e^x$

$y_c = c_1\cos x + c_2\sin x$

$y_p = Ae^x$

(ii) $y'' + y = \cos x$

$y_c = c_1\cos x + c_2\sin x$

$y_p = x(A\cos x + B\sin x)$

(iii) $y'' + y = e^x\cos x + x^2e^x\sin x$

$y_c = c_1\cos x + c_2\sin x$

$y_p = e^x(A\cos x + B\sin x)$
$\quad + e^x(Cx^2 + Dx + E)(F\cos x + G\sin x)$

(iv) $y'' + 2y' + y = e^{-x}$

$y_c = c_1e^{-x} + c_2xe^{-x}$

$y_p = Ax^2e^{-x}$

(v) $y'' + 2y' + y = \sin x + 5 + x + e^x$

$y_c = c_1e^{-x} + c_2xe^{-x}$

$y_p = (A\cos x + B\sin x) + Cx + De^x + E$

$y'' + 4y = 0$ 2a, 2b

$r^2 + 4 = 0$

$r = \pm 2i$

$y = e^{0x}(c_1\cos 2x + c_2\sin 2x)$

i. $y(0) = 0,\ y(\pi) = 1$ doesn't work, discrepancy with c_1 – no solution

ii $y(0) = 0,\ y(\pi) = 0$, $c_1 = 0$, c_2 undetermined – too many solutions

ii $y(0) = 0,\ y(\pi/4) = 1$, $y(0) = c_1 = 0,\ y(\pi/4) = c_2 = 1$, $y = \sin 2x$ – only one :(

3.a) $y' + 2xy = y$

$y' + (2x-1)y = 0$

$y' = (1-2x)y$

$\int \frac{dy}{y} = \int (1-2x)\,dx$

$\log|y| = x - x^2 + c$

$y = \pm e^{x-x^2+c} \Rightarrow y = Ae^{x-x^2}$ (A takes any value)
(check $y=0$.)

b) $y - 2xy = 3x^2 e^{x^2}$

$ye^{\int -2x\,dx} = \int 3x^2 e^{x^2} \cdot e^{\int -2x\,dx}$

$ye^{-x^2} = x^3 + c$

$y = x^3 e^{x^2} + c$

$y(0) = 3$

$3 = 0 \cdot e^0 + c$

$c = 3$

$\Rightarrow y = x^3 e^{x^2} + 3$

3c.) $y'' - 2y' + y = \cos x + 3\sin x$

$r^2 - 2r + 1 = 0$

$(r-1)^2$

$r = 1$

$y_c = c_1 e^x + c_2 x e^x$

$y_p = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

$\cancel{-A\cos x} - \cancel{B\sin x} - 2(-A\sin x + B\cos x) + \cancel{A\cos x} + \cancel{B\sin x}$

$2A\sin x = 3\sin x \longrightarrow A = \frac{3}{2}$

$-2B\cos x = \cos x \longrightarrow B = -\frac{1}{2}$

$$\boxed{y = c_1 e^x + c_2 x e^x + \frac{3}{2}\cos x - \frac{1}{2}\sin x}$$

3. (d) $x^2y' = x^2 + y^2 + yx$

$y' = 1 + \frac{y^2}{x^2} + \frac{y}{x}$

Let $v = \frac{y}{x}$

$\frac{dy}{dx} = 1 + v^2 + v$

$\frac{dv}{dx} = \frac{\frac{dy}{dx}x - y}{x^2}$

$x^2 \frac{dv}{dx} = \frac{dy}{dx}x - y$

$x^2 \frac{dv}{dx} + y = \frac{dy}{dx}x$

$x \frac{dv}{dx} + \frac{y}{x} = \frac{dy}{dx}$

$x \frac{dv}{dx} + v = \frac{dy}{dx}$

$x \frac{dv}{dx} + \cancel{v} = 1 + v^2 \cancel{+v}$

$x \frac{dv}{dx} = 1 + v^2$

$\int \frac{1}{1+v^2}\, dv = \int \frac{1}{x}\, dx$

$\tan^{-1} v = \ln|x| + C$

$v = \tan(\ln|x| + c)$

$\frac{y}{x} = \tan(\ln|x| + c)$

$\boxed{y = x \tan(\ln|x| + c)}$

3 d) $x^2y' = x^2 + y^2 + yx$ (Hint: $v = \frac{y}{x}$)

$v = \frac{y}{x}$, $v' = \frac{dy}{dx} x^{-1} + y(-x^{-2})$

$= \frac{dy}{dx} x^{-1} - yx^{-2}$

$y' = 1 + \frac{y^2}{x^2} + \frac{y}{x}$

$\frac{dy}{dx} = (v' + yx^{-2})x$

$= v'x + yx^{-1}$

$v'x + yx^{-1} = 1 + v^2 + v$

$v'x + v = 1 + v^2 + v$

$v'x = 1 + v^2$

$v' = \frac{1+v^2}{x}$

$\frac{dv}{dx} = \frac{1+v^2}{x}$

$\int \frac{dv}{1+v^2} = \int \frac{1}{x} dx$

$\tan^{-1} v = \ln|x| + c$

$\tan^{-1} \frac{y}{x} = \ln|x| + c$

taking tan on both sides

$\frac{y}{x} = \tan(\ln|x| + c)$

$\boxed{y = x\tan(\ln|x| + c)}$

e) $y' - x = x \sin y$

$y' = x(1 + \sin y)$

$\frac{dy}{dx} = x(1 + \sin y)$

$\int \frac{dy}{1+\sin y} = \int x \, dx$

$\int \frac{dy}{1+\sin y} = \frac{x^2}{2} + c$

$\int \frac{dy}{1+\sin y} = \int \frac{(1-\sin y)}{(1+\sin y)(1-\sin y)} dy$

$= \int \frac{1-\sin y}{1-\sin^2 y} dy = \int \frac{1}{\cos^2 y} - \frac{\sin y}{\cos^2 y} dy$

$= \int \sec^2 y \, dy - \int \sec y \tan y \, dy$

$= \tan y - \sec y$

So, $\tan y - \sec y = \frac{x^2}{2} + c$ (this is as reduced as possible.)

f) This one is surprisingly difficult.

$$xy' + x = e^{y/x} + y$$

First substitute $v = \frac{y}{x}$

$$\Rightarrow v' = \frac{xy' - y}{x^2} \Rightarrow x^2 v' = xy' - y$$

$$\Rightarrow y' = xv' + \frac{y}{x} = xv' + v$$

$$\Rightarrow x(xv' + v) + x = e^v + xv$$

$$\Rightarrow x^2 v' + xv + x = e^v + xv$$

$$\Rightarrow x^2 v' + x = e^v$$

This one is also stubborn, so we try to substitute

$$w = e^{-v} \Rightarrow w' = -v' e^{-v}$$

$$\Rightarrow v' = \frac{-w'}{w}.$$

$$\Rightarrow x^2\left(-\frac{w'}{w}\right) + x = \frac{1}{w}$$

$$\Rightarrow -x^2 w' + xw = 1$$

$$\Rightarrow w' - \frac{1}{x} w = \frac{1}{x^2}$$

Now it is in linear form. We take $I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$. and multiply through.

$$\Rightarrow \frac{1}{x} w' - \frac{1}{x^2} w = \frac{1}{x^3}$$

$$\Rightarrow \left(\frac{1}{x} w\right)' = \frac{1}{x^3}$$

$$\Rightarrow \frac{1}{x} w = -\frac{1}{2x^2} + C$$

$$\Rightarrow w = -\frac{1}{2x} + Cx$$

$$\Rightarrow e^{-v} = -\frac{1}{2x} + Cx \Rightarrow -v = \log\left(-\frac{1}{2x} + Cx\right)$$

$$\Rightarrow \frac{y}{x} = -\log\left(-\frac{1}{2x} + Cx\right) \Rightarrow \boxed{y = -x\log\left(-\frac{1}{2x} + Cx\right)}$$

g) $y' + y = xy^3$

$$\Rightarrow y^{-3}y' + y^{-2} = x$$

$$\Rightarrow -2y^{-3}y' - 2y^{-2} = -2x$$

$$\Rightarrow v' - 2v = -2x$$

$$\Rightarrow e^{-2x}v' - 2e^{-2x}v = -2xe^{-2x}$$

$$\Rightarrow (e^{-2x}v)' = -2xe^{-2x}$$

$$\Rightarrow e^{-2x}v = \int -2xe^{-2x}\,dx$$

$$\Rightarrow e^{-2x}v = xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

$$\Rightarrow v = x + \frac{1}{2} + Ce^{2x}$$

$$\Rightarrow y^{-2} = x + \frac{1}{2} + Ce^{2x} \Rightarrow y = \pm\left(x + \frac{1}{2} + Ce^{2x}\right)^{-\frac{1}{2}}$$

This is a Bernoulli equation, so we substitute $v = y^{-2}$.

$$\Rightarrow v' = -2y^{-3}y'.$$

$I(x) = e^{\int -2\,dx} = e^{-2x}$. Multiply through.

<u>Integration by Parts</u>

$$\int -2xe^{-2x}\,dx \qquad \begin{bmatrix} u = -2x & dv = e^{-2x}dx \\ du = -2dx & v = -\frac{1}{2}e^{-2x} \end{bmatrix}$$

$$= xe^{-2x} - \int e^{-2x}\,dx = xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

Britta Nevitt + Lisa Du

4. (a) $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$

$$\frac{dy}{dt} = \left(1\,\frac{kg}{min}\right) - \left(\frac{y(t)}{100-2t}\right)\left(\frac{2L}{min}\right)$$

$$\frac{dy}{dt} = 1 - \frac{2y(t)}{100-2t}$$

$$\frac{dy}{dt} = 1 - \frac{y(t)}{50-t}$$

$$\frac{dy}{dt} + \frac{y(t)}{50-t} = 1$$

$$I(x) = e^{\int \frac{1}{50-t}dt} = e^{-\int \frac{1}{u}du} = e^{-\ln|u|} = e^{-\ln|50-t|} = \frac{1}{|50-t|}$$

$$\left[\begin{array}{l} u = 50-t \\ du = -dt \Rightarrow -du = dt \end{array}\right]$$

$$I(x)\left[\frac{dy}{dt} + \frac{y(t)}{50-t} = 1\right]$$

$$\int \left(\frac{1}{|50-t|}y\right)' = \int \frac{1}{|50-t|}dt$$

$$\frac{y}{|50-t|} = -\ln|50-t| + C$$

$$\boxed{y = (|50-t|)(-\ln|50-t| + C)}$$

(b) $y(0) = 10$ kg

$$10 = (|50-0|)(-\ln|50-0| + C)$$

$$10 = -50\ln 50 + 50C$$

$$\frac{1}{5} = -\ln 50 + C$$

$$C = \frac{1}{5} + \ln 50$$

$$\boxed{y(t) = (|50-t|)\left(-\ln|50-t| + \frac{1}{5} + \ln 50\right)}$$

(5) Population $P(t)$

$$P'(t) = P(t)\ln(P(t))$$

$P(0)=4$

$P(1)=7$

$$\frac{dP}{dt} = KP\ln(P)$$

$$\int \frac{dP}{P\ln(P)} = \int K\,dt$$

(1)

(1) $$\int \frac{dP}{P\ln(P)} = \left[\begin{array}{l} u = \ln P \\ du = \frac{1}{P}\,dP \end{array}\right]$$

$$\int \frac{1}{u}\,du$$

$$\ln|u| = \ln|\ln(P)|$$

$$\ln|\ln(P)| = Kt + C$$

$$\ln(P) = \pm e^{Kt+C}$$

$$\ln(P) = Ae^{Kt} \qquad A = \pm e^{C}$$

$$\boxed{P = e^{Ae^{Kt}}}$$

$$4 = e^{A}$$

$$A = \ln 4$$

$$7 = e^{\ln 4 e^{k}}$$

$$\ln 7 = \ln 4\, e^{k}$$

$$\frac{\ln 7}{\ln 4} = e^{k}$$

$$k = \ln\left(\frac{\ln 7}{\ln 4}\right)$$

$$\boxed{P(t) = e^{(\ln 4)e^{\ln\left(\frac{\ln 7}{\ln 4}\right)t}}}$$