Gauss Elimination

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Guiding Question

Given a matrix equation $Ax = b$, for a real matrix A and a real vector b, how do we find a solution x?

Graphical Method

Consider each equation as defining a line / plane / hyperplane in Euclidean space.

Draw each plane and see where they all intersect.

Cramer's Rule

For $Ax = b$, let $A_j =$ the matrix formed by replacing the j-th column of A with b. Then:

$$x_j = \frac{\det(A_j)}{\det(A)}.$$

What is the determinant of a matrix? See next slide.

Determinants

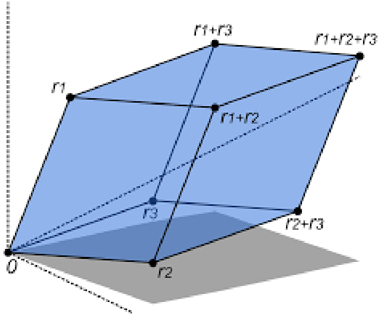
The determinant of a square matrix A can be defined inductively as follows:

1. The determinant of a 1×1 matrix is its only entry.
2. The determinant of a $n \times n$ matrix can be obtained by: Fixing a row j. Defining the matrix $A(j, k)$ to be the matrix leaving out the j-th row and k-th column.

$$\det(A) = \sum_{k=1}^{n} (-1)^{j+k} A_{j,k} \det(A(j, k))$$

Determinants – Geometric View

If you consider the matrix A as a linear map between two vector spaces, then the determinant can be thought of as the volume of the unit cube after passing through the map A.



–image from Wikipedia.

Naive Gauss Elimination

INPUT: $n \times (n+1)$ Matrix $A|b$

1. Fix pivot variable $A_{1,1}$.
2. Perform **row operations** to eliminate everything below the pivot variable.
3. If $k < n$, change pivot variable from $A_{k,k}$ to $A_{k+1,k+1}$ and repeat step 2.
4. Solve for x_k and substitute its value into Row $k-1$. Repeat to solve for each x_k.

Row operations: Permute rows, add a scalar multiple of row j to row k.

Naive Gaussian Elimination - Example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 \\ 3 & 1 & 4 & 2 \end{array}\right)$$

Naive Gauss - Complexity

How many operations does Naive Gaussian elimination take?

$$
\begin{aligned}
\#\text{operations} &= \sum_{k=1}^{n-1} \#(\text{operations at pivot } k) \\
&+ \sum_{k=1}^{n} \#(\text{operations to solve for } x_k) \\
&= \sum_{k=1}^{n-1} 2(n-k)(n-k-1) + \sum_{k=1}^{n} (n-k+2)
\end{aligned}
$$

Naive Gauss - Complexity

$$
\begin{aligned}
&= \sum_{k=1}^{n-1} 2n^2 - \sum_{k=1}^{n-1} 4nk + \sum_{k=1}^{n-1} 2k^2 - \sum_{k=1}^{n-1} 2n \\
&+ \sum_{k=1}^{n-1} 2k + \sum_{k=1}^{n} n - \sum_{k=1}^{n} k + \sum_{k=1}^{n} 2. \\
&= 2n^3 - 3n^2 + 3n + (-4n+1)\sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1} k^2.
\end{aligned}
$$

Note that $\sum_{k=1}^{n-1} k = \frac{1}{2}n^2 + O(n)$ and $\sum_{k=1}^{n-1} k^2 = \frac{1}{3}n^3 + O(n^2)$.

Naive Gauss - Complexity

So we are left with:

$$
\begin{aligned}
&= 2n^3 - 3n^2 + 3n + (-4n+1)(\frac{1}{2}n^2 + O(n)) + 2(\frac{1}{3}n^3 + O(n^2)) \\
&= \frac{2}{3}n^3 + O(n^2).
\end{aligned}
$$

Pivoting

We have to be careful! Sometimes $A_{k,k} = 0$ or is numerically very close to zero.

- Partial pivoting means at each step, search column k for the largest element A_{jk} then switch rows j and k.
- Complete pivoting means that we also search for the highest value in the row j (This would mean relabeling variables, which is undesirable).

Pivoting - Example

Example

$$\begin{pmatrix} .0003 & 3.0000 \\ 1.0000 & 1.0000 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.0001 \\ 1.0000 \end{pmatrix}$$

Compare the results you obtain via naive Gaussian elimination, and pivoting. (Use MATLAB for your computation.)

Tridiagonal Systems

A tridiagonal system is a matrix equation $Ax = b$, when the only nonzero entries of A are $A_{i,i-1}, A_{i,i}$, and $A_{i,i+1}$ (wherever these indices make sense).

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{n-1,n} & a_{n,n} \end{pmatrix}$$

Tridiagonal Systems - Complexity

Question: How many operations does Gaussian Elimination take for a tridiagonal system?

Perform a complexity computation similar to the slides above regarding Naive Gaussian Elimination, accounting for both elimination and substitution steps. Your result should be $O(n)$.

LU Factorization

Problem: Find a way to quickly solve $Ax = b$ when A is fixed but we want to solve for a number of inputs b.

Idea: Factor A as $A = LU$ where L is lower-triangular and U is upper-triangular. Then, suppose there exists a vector d so that $Ld = b$.

$$Ax = b \implies LUx = b \implies LUx = Ld \iff Ux = d.$$

At this point, we have two equations with triangular matrices:

$$Ux = d, Ld = b.$$

Solving with LU Factorization

Given these two triangular systems $Ux = d, Ld = b$, we do not need to worry about the elimination part of Gaussian elimination – just the substitution part.

Solve for d_1 through d_n substituting down at each step.
Then solve for x_n through x_1 substituting up at each step.
This drastically cuts down complexity.

LU Factorization – How?

It should be clear by now that an LU factorization would be really useful for equation solving. But how do we obtain it?
Fix A as our matrix.
Perform the standard Gaussian elimination, when our pivot variable is A_{kk}, we eliminate entry A_{jk} below the diagonal by replacing row R_j with $c_{jk} * R_k + R_j$. Set L_{jk} to be $-c_{jk}$.

- U is the matrix spit out by Gaussian elimination.
- L is the matrix with ones on the diagonal, and the lower entries obtained as above.

LU Factorization - Example

Let A be the matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 8 \\ 3 & 1 & 4 \end{pmatrix}$$

Let b be the vector $(4, 7, 2)$. Solve $Ax = b$ by:

1. Standard Gaussian Elimination.
2. Finding $A = LU$ then solving $Ld = b$ and $Ux = d$.