Cholesky Factorization

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LU Factorization

Any matrix A, after permuting the rows of A with a permutation matrix P, can be factorized as $PA = LU$ where:

L is lower-triangular, and
U is upper-triangular.

The MATLAB command `[L,U,P] = lu(A)` obtains the LU Factorization.

LU Factorization - Example

We define the following matrix:

$$A = \begin{bmatrix} 7.102 & 3.431 & 5.528 \\ 8.412 & 2.135 & 1.709 \\ 4.047 & 5.210 & 8.227 \end{bmatrix}$$

The command `[L,U] = lu(A)` gives an upper-triangular and "lower-triangular" matrix whose product is A.

Changing the pivot options can sometimes give an "honest" lower-triangular matrix.

LU for Symmetric Matrices

Recall: A symmetric matrix A is a matrix whose entries $a_{ij} = a_{ji}$ for all i, j. Equivalently, $A = A^T$.
The LU factorization for A can be $A = U^T U$. I.e. the lower-triangular matrix is the transpose of the upper-triangular matrix. This factorization is called the *Cholesky factorization*.

Andre-Louis Cholesky

Cholesky factorization - example

$$A = \begin{bmatrix} 7.102 & 3.431 & 5.528 \\ 3.431 & 2.135 & 1.709 \\ 5.528 & 1.709 & 8.227 \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & 0 & 0 \\ u_{21} & u_{22} & 0 \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ 0 & u_{22} & u_{32} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Note that $a_{11} = u_{11}^2$, so set $u_{11} = \sqrt{7.102}$.
Moving to the next row, $u_{21}u_{11} = a_{21}$ so set
$u_{21} = 3.431/7.102 = .4831$.
We know $a_{22} = u_{21}^2 + u_{22}^2$ so solve:
$u_{22} = \sqrt{a_{22} - u_{21}^2} = \sqrt{2.135 - .4831^2} = 1.379$. Etc.

General formula for Cholesky Entries

The general formula for diagonal entries is:

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}.$$

The general formula for off-diagonal entries is:

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}}$$

Cholesky in MATLAB

In MATLAB, the Cholesky decomposition of a matrix A is obtained using the command `U = chol(A)`.