Matrix Inverses

Zvi Rosen
Department of Mathematics

October 21, 2016

Matrix Inverse - Definition

Definition

Let A be an $n \times n$ matrix. The inverse of A, denoted A^{-1} is the matrix which satisfies

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n.$$

Note that in general $AB \neq BA$ for matrices, but for the inverse matrix $AA^{-1} = A^{-1}A$.

Matrix Inverse - Example

Define the matrix

$$M = \begin{pmatrix} 1 & 0 & 3 \\ 6 & -2 & 8 \\ -1 & 1 & 4 \end{pmatrix}$$

Using the MATLAB command `inv(M)`, we obtain:

$$M^{-1} = \begin{pmatrix} 4 & -0.75 & -1.5 \\ 8 & -1.75 & -2.5 \\ -1 & 0.25 & 0.5 \end{pmatrix}$$

Algorithm for Computing Inverses

Note that the inverse matrix takes input $(1,0,0)^T$ and gives output the solution to $Ax_1 = (1,0,0)^T$, similarly,

$$(0,1,0)^T \to \text{ solution of } Ax_2 = (0,1,0)^T$$

$$(0,0,1)^T \to \text{ solution of } Ax_3 = (0,0,1)^T$$

This will be enough to determine our inverse matrix

$$A^{-1} = \left[\begin{array}{c|c|c} x_1 & x_2 & x_3 \end{array} \right].$$

Vector Norms

A *vector norm* is a mapping from the set of vectors to the set of nonnegative real numbers. It has the following properties:

1. $||a\mathbf{v}|| = |a| \cdot ||\mathbf{v}||$ for real scalars a,
2. $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ (triangle inequality), and
3. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v}$ is the zero vector.

$||v||_e$: The Euclidean norm.
$||v||_p$: The p-norm, for $p \geq 1$.
$||v||_1$: The 1-norm.
$||v||_\infty$: The infinity-norm.

Matrix Norms

A *matrix norm* is a mapping from the set of matrices to the set of nonnegative real numbers, with all the same properties as the vector norm.

A matrix norm is induced by a vector norm if you define it as $||A|| = \max_{x \in \mathbb{R}^n} ||Ax||/||x||$, using the vector norm on the right-hand side. In each of the following,

$||M||_f$: The Frobenius norm. (not induced)

$||M||_1$: The column-sum norm. (induced by vector 1-norm)

$||M||_\infty$: The row-sum norm. (induced by vector ∞-norm)

$||M||_2$: The spectral norm. (induced by vector Euclidean norm)

The Spectral Norm

The spectral norm is the square root of the largest eigenvalue of $A^T A$.

Condition Number

Definition

The **condition number** of a function is a measurement of how much the output changes relative to small changes in the input.

Definition

The **matrix condition number** of A is the condition number of the function which takes b as input and then solves for $Ax = b$.
Precisely, if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$, it is equal to

$$\max_{b, \Delta b} \frac{||\Delta x||/||x||}{||\Delta b||/||b||}.$$

Matrix Condition Number

Let's begin from this relationship:

$$A(x+\Delta x) = b+\Delta b \iff A(\Delta x) = \Delta b \iff \Delta x = A^{-1}\Delta b.$$

This means that:

$$\frac{||\Delta x||/||x||}{||\Delta b||/||b||} = \frac{||A^{-1}\Delta b||/||A^{-1}b||}{||\Delta b||/||b||} = \frac{||A^{-1}\Delta b||}{||\Delta b||} \cdot \frac{||b||}{||A^{-1}b||}.$$

Substitute $b = Ac$ for some vector c.

$$= \frac{||A^{-1}\Delta b||}{||\Delta b||} \cdot \frac{||Ac||}{||c||}.$$

Maximizing over Δb and c gives the product $||A^{-1}|| \cdot ||A||$, using the operator norm.

Condition number - Example

$$M = \begin{pmatrix} 1 & 0 & 3 \\ 6 & -2 & 8 \\ -1 & 1 & 4 \end{pmatrix}, M^{-1} = \begin{pmatrix} 4 & -0.75 & -1.5 \\ 8 & -1.75 & -2.5 \\ -1 & 0.25 & 0.5 \end{pmatrix}$$

norm(M) $\cdot$ norm(M^{-1}), computed in MATLAB, gives 105.6137. The matrix is not well-conditioned.