Let a and b be positive numbers with $\mathbf{a > b}$. Let $\mathbf{a_1}$ be their arithmetic mean and $\mathbf{b_1}$ their geometric mean:

$$\mathbf{a_1 = \frac{a+b}{2} \qquad b_1 = \sqrt{ab}}$$

Repeat this process so that, in general,

$$\mathbf{a_{n+1} = \frac{a_n+b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}}$$

a) Use mathematical induction to show that

$$\mathbf{a_n > a_{n+1} > b_{n+1} > b_n}.$$

As with all induction arguments, we need a base case and an induction step.

1. Base Case

We start with the base case $n = 1$. We need to prove that $a_1 > a_2 > b_2 > b_1$. Equivalently, we want to show that $a_1 > \frac{a_1+b_1}{2} > \sqrt{a_1 b_1} > b_1$

First we demonstrate that $a_1 > b_1$. We know that $(\sqrt{a} - \sqrt{b})^2 > 0$ since $a > b$.

$$(\sqrt{a} - \sqrt{b})^2 > 0 \Rightarrow a - 2\sqrt{a}\sqrt{b} + b > 0 \Rightarrow a + b > 2\sqrt{a}\sqrt{b} \Rightarrow \frac{a+b}{2} > \sqrt{ab} \tag{1}$$

Using this fact, we can show:

$$a_1 > b_1 \Rightarrow 2a_1 > a_1 + b_1 \Rightarrow a_1 > \frac{a_1+b_1}{2} \Rightarrow a_1 > a_2. \tag{2}$$

$$a_1 > b_1 \Rightarrow a_1 b_1 > b_1^2 \text{ (true, because } b_1 > 0) \Rightarrow \sqrt{a_1 b_1} > b_1 \Rightarrow b_2 > b_1. \tag{3}$$

Finally, we use a variation on argument (1) to show that $a_2 > b_2$:

$$(\sqrt{a}_1 - \sqrt{b}_1)^2 > 0 \Rightarrow a_1 - 2\sqrt{a}_1\sqrt{b}_1 + b_1 > 0 \Rightarrow \frac{a_1+b_1}{2} > \sqrt{a_1 b_1} \Rightarrow a_2 > b_2 \tag{4}$$

Putting these together, we find that:

$$a_1 > a_2 > b_2 > b_1.$$

2. Induction Step

We now proceed to the induction step. This is where we assume that $a_n > a_{n+1} > b_{n+1} > b_n$, and we need to prove that $a_{n+1} > a_{n+2} > b_{n+2} > b_{n+1}$. These arguments are going to be similar to the ones in the previous step:

$$a_{n+1} > b_{n+1} \Rightarrow 2a_{n+1} > a_{n+1} + b_{n+1} \Rightarrow a_{n+1} > \frac{a_{n+1}+b_{n+1}}{2} \Rightarrow a_{n+1} > a_{n+2}. \tag{5}$$

$$a_{n+1} > b_{n+1} \Rightarrow a_{n+1} b_{n+1} > b_{n+1}^2 \Rightarrow \sqrt{a_{n+1} b_{n+1}} > b_{n+1} \Rightarrow b_{n+2} > b_{n+1}. \tag{6}$$

$$(\sqrt{a}_{n+1} - \sqrt{b}_{n+1})^2 > 0 \Rightarrow a_{n+1} - 2\sqrt{a}_{n+1}\sqrt{b}_{n+1} + b_{n+1} > 0 \tag{7}$$

$$\Rightarrow \frac{a_{n+1}+b_{n+1}}{2} > \sqrt{a_{n+1} b_{n+1}} \Rightarrow a_{n+2} > b_{n+2}.$$

This gives us the result:

$$a_{n+1} > a_{n+2} > b_{n+2} > b_{n+1}.$$

So, by induction, we proved that, for all n,

$$a_n > a_{n+1} > b_{n+1} > b_n.$$

b) Deduce that both $\{\mathbf{a_n}\}$ and $\{\mathbf{b_n}\}$ are convergent.

Both a_n and b_n are bounded above by a_1 and below by b_1. The sequence a_n is monotone decreasing, and the sequence b_n is monotone increasing. Therefore, by the Monotone Sequence Theorem, both sequences converge.

c) Show that $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \lim_{\mathbf{n}\to\infty} \mathbf{b_n}$. Gauss called the common value of these limits the arithmetic-geometric mean of he numbers a and b.

Let $A = \lim_{n\to\infty}\{a_n\}$ and $B = \lim_{n\to\infty}\{b_n\}$. We can take either recurrence relation, take the limit as $n \to \infty$, and we will find that $A = B$.

Starting with the recurrence relation for a_n:

$$\begin{gathered} \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + b_n}{2} \\ \Rightarrow A = \frac{A+B}{2} \\ \Rightarrow \frac{A}{2} = \frac{B}{2} \\ \Rightarrow A = B. \end{gathered}$$

Starting with the recurrence relation for b_n:

$$\begin{gathered} \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \sqrt{a_n b_n} \\ \Rightarrow B = \sqrt{AB} \\ \Rightarrow B^2 = AB \\ \Rightarrow B = A. \end{gathered}$$