Quiz 3

MATH 1B, SPRING 2012

February 10, 2012

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SECTION: NAME:

Solve the following integral, showing all steps clearly:

$$\int_0^\infty \frac{x \arctan(x)\, dx}{(1+x^2)^2}$$

Be careful about identifying improper integrals, and expressing them as limits of proper integrals!

Approach 1 (Trig Sub)

First, we note that the integral is improper, so we express it as: $\lim_{t\to\infty} \int_0^t \frac{x \arctan(x)\, dx}{(1+x^2)^2}$.

$\left[\text{Let } x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta \quad 1+x^2 = 1+\tan^2\theta = \sec^2\theta. \right]$

$$= \lim_{t\to\infty} \int_0^{\arctan t} \frac{\theta \tan\theta}{\sec^4\theta} \sec^2\theta\, d\theta$$

$$= \lim_{t\to\infty} \int_0^{\arctan t} \theta \sin\theta \cos\theta\, d\theta = \lim_{t\to\infty} \int_0^{\arctan t} \theta \cdot \tfrac{1}{2}\sin 2\theta\, d\theta.$$

$$\begin{bmatrix} u = \frac{1}{2}\theta & dv = \sin 2\theta \\ du = \frac{1}{2}d\theta & v = -\frac{1}{2}\cos 2\theta \end{bmatrix}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{4}\theta \cos 2\theta \Big|_0^{\arctan t} + \frac{1}{4}\int_0^{\arctan t} \cos 2\theta\, d\theta \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{4}\arctan t \cdot \cos(2\tan^{-1}(t)) + \frac{1}{8}\sin(2\arctan t) \right]$$

$$= -\frac{1}{4}\cdot\frac{\pi}{2}\cdot(-1) + \frac{1}{8}\sin\pi = \frac{\pi}{8}.$$

Approach 2 (IBP first)

$$\int_0^\infty \frac{x \arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{x \arctan x \, dx}{(1+x^2)^2}.$$ (the limit of a proper integral)

IBP: $\left[\begin{array}{ll} u = \arctan x & dv = \dfrac{x}{(1+x^2)^2} \\ du = \dfrac{dx}{1+x^2} & v = \displaystyle\int \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{dw}{w^2} = -\frac{1}{2w} = -\frac{1}{2(1+x^2)} \end{array}\right]$

(u-sub: $w = 1+x^2$)

$$= -\frac{\arctan x}{2(1+x^2)}\Bigg|_0^t + \frac{1}{2}\int_0^t \frac{dx}{(1+x^2)^2}.$$

Let $x = \tan\theta$. $1+x^2 = \sec^2\theta$.
$dx = \sec^2\theta\, d\theta$

$$= -\frac{\arctan t}{2(1+t^2)} + \frac{1}{2}\int_0^{\arctan t} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \frac{-\arctan t}{2(1+t^2)} + \frac{1}{2}\int_0^{\arctan t} \cos^2\theta\, d\theta$$

$$= \frac{-\arctan t}{2(1+t^2)} + \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right]_0^{\arctan t} = \frac{-\arctan t}{2(1+t^2)} + \frac{1}{4}\arctan t + \frac{1}{8}\sin\left(2\tan^{-1}(t)\right)$$

Taking the limit as $t\to\infty$, we find $\arctan t \to \frac{\pi}{2}$.

$$= 0 + \frac{\pi}{8} + 0 = \frac{\pi}{8}.$$