Below are a few tips and tricks that may be useful for the midterm. Good luck studying!

1. Sequences & Limits

(1) Sequences with fractions or products: Try L'Hospital's Rule.
Example: Find the limit of $\{ne^{-n}\}$.

(2) Sequences with exponents: Remember that $\log(\lim_{n\to\infty} a_n) = \lim_{n\to\infty}(\log a_n)$ as long as a_n converges to some positive value.
Example: Find the limit of $\left\{\left(\frac{1+n}{2+n}\right)^n\right\}$.

(3) If there is a recursively defined sequence, try to show that it is monotone and bounded using induction.
Example: Define the sequence $\{a_n\}$ by:

$$a_1 = 1, a_{n+1} = \sqrt[3]{a_n + 6}.$$

Prove that the sequence converges, and find the limit.

(4) If many functions are involved, try expanding them into Taylor Series, and canceling terms.
Example: Find the limit of $\dfrac{\cos x - 1}{e^{x^2} - 1}$ as $x \to 0$.

2. Series

(1) Remember that the "Test for Divergence" is *only* good for divergence – it cannot show that a series converges.
Example: $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges by the Integral Test (check: positive, decreasing, continuous); however, the terms do converge to 0.

(2) Check for the requirements of any convergence tests you plan on using, and make a note of them.
Example: For the Alternating Series Test, check that the sequence of terms $a_n = (-1)^n b_n$ where (1) $b_n > 0$ for all n, (2) $b_n \geq b_{n+1}$, and (3) $\lim b_n = 0$.

(3) When you are looking for a series to compare to using the Limit Comparison Test, look at the general term of the series for a hint.
Example: Determine whether this series converges: $\displaystyle\sum_{n=1}^{\infty}[2^{1/n} - 1]$. Use Limit Comparison Test with $b_n = \frac{1}{n}$.

(4) If you are asked to find the sum of an infinite series, look for geometric series, telescoping sums, and Taylor Series evaluated at a point, or perhaps a combination of those.
Example: Evaluate the sum $\sum_{n=0}^{\infty}\left(1 + \frac{(-1)^n}{2n+1}\right)\left(\sqrt{\frac{1}{3}}\right)^{2n+1}$.

3. Taylor Series

(1) Given a composite function $f(g(x))$, you can plug in $g(x)$ into the Taylor series for $f(u)$.
Example: $\cos(x^3/2) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^3/2)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(2n)!}x^{6n}$.

(2) The n-th derivative of $f(x)$ shows up in the coefficient of the x^n term, even if the value of the index is different.

Example: In the Taylor Series above, $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n)!} x^{6n}$, the value of $f^{(4)}(0)$ is not in the $n = 4$ term, rather in the coefficient of x^4; i.e. 0.

(3) When using the Alternating Series Estimation Theorem to estimate error, make sure that the series satisfies all the requirements of the Alternating Series Test.

(4) When calculating error using Taylor's Inequality, find the maximum of the $(n+1)$-th the derivative on the entire interval, not only the center or endpoints.

(5) When trying to find closed-form functions from series, use the following tips:
   (a) Figure out which known series you're aiming for. Factorials are a big hint.
   (b) If x is not in the expression, figure out what x the series is evaluated at - look for terms with an exponent containing the index (n).
   (c) An extra n in the numerator comes from derivatives; in the denominator, it comes from integration.
   (d) Once you figure out the series you should start from, equate the series to a function, and manipulate both sides.

4. Extra Practice Problems

(1) What is the interval of convergence of the series $\sum_{n=1}^{\infty} n(n+1)^2(\pi x - 2)^n$?

(2) The approximation $e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$ is obtained from the first four terms of the MacLaurin expansion of e^x, at $x = 1$. From the Taylor Remainder Theorem, what is the guaranteed maximum absolute value of the error, $|R_n|$, of this approximation?

(3) What is the third nonzero term of the MacLaurin series of $f(x) = \int_0^x \sin(t^2)\mathrm{d}t$?

(4) Find the following sums

$$\sum_{n=0}^{\infty} \frac{nx^n}{2^n n!} \qquad\qquad \sum_{n=0}^{\infty} \frac{n}{2^n}$$

$$\sum_{n=0}^{\infty} n(n+1)x^n \qquad\qquad \sum_{n=0}^{\infty} \frac{1}{2^n(n+1)}$$