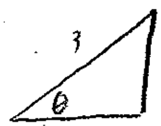
1.) a.) $\int \frac{x}{\sqrt{9-x^2}}\,dx$ sub in $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$

$= \int \frac{9\sin\theta\cos\theta\,d\theta}{\sqrt{9-(3\sin x)^2}}$ $\sqrt{9-(3\sin x)^2} = \sqrt{9-9\sin^2 x} = \sqrt{9(1-\sin^2 x} = \sqrt{9\cos^2 x} = 3|\cos x|$

$= \int \frac{9\sin\theta\cos\theta\,d\theta}{3|\cos\theta|}$ $|\cos\theta| = \cos\theta$ if $0 \le \theta \le \frac{\pi}{2}$

$= \frac{9}{3}\int \frac{\sin\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$

$= 3\int \sin\theta\,d\theta$



$\sin\theta = \frac{x}{3}$ so $a^2 + x^2 = 9$

$a = \sqrt{9-x^2}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= 3(-\cos\theta) + C$

$= -3\,\frac{\sqrt{9-x^2}}{3} + C$

$\boxed{= -\sqrt{9-x^2} + C}$

b.) $\int \frac{x}{\sqrt{9-x^2}}\,dx$ $u = 9-x^2$ and $du = -2x\,dx$

$= -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{9-x^2}}$

$= -\frac{1}{2}\int \frac{du}{u^{1/2}}$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{\cancel{2}}(\cancel{2}u^{1/2}) + C$

$= -\sqrt{u} + C$

$\boxed{= -\sqrt{9-x^2} + C}$

c.) The answers are the same. Yea!

(29)

$$\int \frac{dx}{x^2\sqrt{16-x^2}}$$

$x = 4\sin\theta$
$dx = 4\cos\theta \, d\theta$

$$= \int \frac{4\cos\theta \, d\theta}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}$$

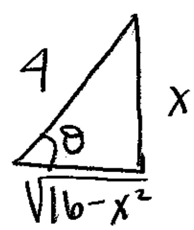
$$= \frac{1}{4}\int \frac{\cos\theta \, d\theta}{\sin^2\theta(4\cos\theta)}$$

$$= \frac{1}{16}\int \frac{\cancel{\cos}\theta \, d\theta}{\sin^2\theta \, \cancel{\cos}\theta}$$

$$= \frac{1}{16}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{16}\int \csc^2\theta \, d\theta$$

$$= \frac{1}{16}(-\cot\theta)$$



$\theta = \sin^{-1}\frac{x}{4}$

$$= \frac{1}{16}\left(-\cot\left(\sin^{-1}\frac{x}{4}\right)\right)$$

$$= \frac{1}{16}\left(-\frac{\sqrt{16-x^2}}{x}\right)$$

$$= \boxed{\frac{-\sqrt{16-x^2}}{16x} + C}$$

2) b) $\displaystyle\int \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

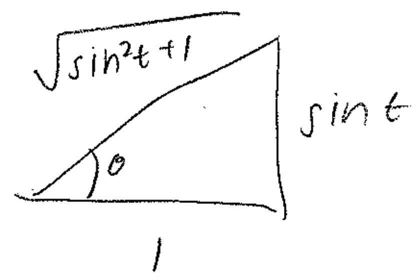
$\sin t = \tan\theta$
$\cos t\,dt = \sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{1+\tan^2\theta}}\cdot\sec^2\theta\,d\theta \Rightarrow \int \sec\theta\,d\theta \Rightarrow$$

$$\ln|\sec\theta+\tan\theta|+C$$

$$\ln|\sec\theta+\sin t|+C$$

$$\boxed{\ln\left|\sqrt{\sin^2 t+1}+\sin t\right|+C}$$



3\. a) False. Let $x = 8\pi$.

$\sin(8\pi) = 0$. $\sin^{-1}(0) = 0 \neq 8\pi$.

b) False. Let $x = -\frac{\pi}{4}$. $\sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$.

Then, $\sqrt{(\sqrt{2})^2 - 1} = 1$.

On the other hand, $\tan\left(-\frac{\pi}{4}\right) = -1 \neq +1$.

c) True. On its domain, $\sin^{-1}(x)$ gives a θ that returns only this x as its sine.

1a. $\dfrac{x^3+2x-1}{x^2(x-4)(x+1)} = \left[\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4} + \dfrac{D}{x+1}\right]$

b. $\dfrac{3x^2-7x}{(x-2)^2(x^2+x+1)^2x^3} = \left[\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x-2} + \dfrac{E}{(x-2)^2} + \dfrac{Fx+G}{x^2+x+1} + \dfrac{Hx+I}{(x^2+x+1)^2}\right]$

c.
$$\begin{array}{r|l} & x \quad 1 \\ \hline x^2-x & x^3+0x^2+0x-1 \\ & \underline{x^3-x^2} \\ & \qquad x^2 \\ & \qquad \underline{x^2-x} \\ & \qquad\quad x-1 \end{array}$$

$\dfrac{x^3-1}{x^2-x} = x+1+\dfrac{x-1}{x^2-x} = \left[x+1+\dfrac{A}{x}+\dfrac{B}{x-1}\right]$

OR

$x+1+\dfrac{1}{x} = \left[x+1+\dfrac{A}{x}\right]$

Partial Fractions

2. a) $\int \frac{dx}{x^2-4} = \int \frac{dx}{(x+2)(x-2)}$

$$\left(\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}\right) x^2-4$$

$$1 = A(x-2) + B(x+2)$$

$$(A+B)x - 2A + 2B$$

$$(A+B=0)2$$

$$2B-2A=1$$

$$+\ 2A+2B=0$$

$$4B=1$$

$$B=\frac{1}{4}$$

$$A=-\frac{1}{4}$$

$$= \int \left(\frac{-1}{4(x+2)} + \frac{1}{4(x-2)}\right) dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

b) $\int \frac{dx}{x^2-4}$

$$x = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta\, d\theta$$

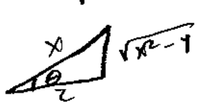
$$= \int \frac{2\sec\theta \cancel{\tan\theta}}{(4\sec^2\theta - 4)} d\theta$$

$$= \int \frac{\cancel{2}\sec\theta \cancel{\tan\theta}}{2\cancel{4}\tan^{\cancel{2}}\theta} d\theta$$

$$= \int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta} \cdot \frac{1}{2}\, d\theta$$

$$= \int \frac{1}{2}\csc\theta\, d\theta$$

$$= \frac{1}{2}\ln|\csc\theta - \cot\theta| + C$$



$$= \frac{1}{2}\ln\left|\frac{x}{\sqrt{x^2-4}} - \frac{2}{\sqrt{x^2-4}}\right| + C$$

$$\frac{1}{2}\ln\left|\frac{x-2}{\sqrt{x^2-4}}\right| + C$$

2. c) $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C_1 \stackrel{?}{=} \frac{1}{2}\ln\left|\frac{x-2}{\sqrt{x^2-4}}\right| + C_2.$

$$\text{RHS} = \frac{1}{2}\ln\left|\frac{x-2}{\sqrt{(x+2)(x-2)}}\right| + C_2$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x-2}}{\sqrt{x+2}}\right| + C_2$$

$$= \frac{1}{2}\ln\left|\left(\frac{x-2}{x+2}\right)^{1/2}\right| + C_2$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C_2 = \text{LHS}. \checkmark$$

Question 3A - Partial Fractions

$$\int \frac{6x^3 + 7x^2 - 2x - 5}{x^4 - x^2}$$

$$= \int \frac{6x^3 + 7x^2 - 2x - 5}{x^2(x^2 - 1)} = \int \frac{6x^3 + 7x^2 - 2x - 5}{x^2(x+1)(x-1)}$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x-1)}$$

$$Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + Dx^3 + Dx^2$$

$$\begin{aligned} A + C + D &= 6 \\ B + D - C &= 7 \\ -A &= -2 \\ -B &= -5 \end{aligned} \implies \begin{aligned} A &= 2 \\ B &= 5 \\ D &= 3 \\ C &= 1 \end{aligned}$$

$$\int \frac{2}{x} + \int \frac{5}{x^2} + \int \frac{1}{(x+1)} + \int \frac{3}{(x-1)}$$

$$\boxed{= 2\ln|x| - \frac{5}{x} + \ln|x+1| + 3\ln|x-1| + C}$$

3.(b)

$$\int \frac{3 \cdot e^{2t}}{e^{2t} - e^{t} - 6} \cdot dt$$

$u = e^t$

$du = e^t . dt$

$= u \cdot dt$

$dt = \frac{du}{u}$

$$= \int \frac{3 \cdot \cancel{u}}{u^2 - u - 6} \cdot \frac{du}{\cancel{u}}$$

$$\frac{3u}{u^2 - u - 6} = \frac{A}{u+2} + \frac{B}{u-3}$$

$$3u = A(u-3) + B(u+2)$$

$$= (A+B)u + (-3A + 2B)$$

$A + B = 3$ $\quad A = 3 - B$

$-3A + 2B = 0$ $\quad -9 + 3B + 2B = 0$

$5B = 9. \quad B = \frac{9}{5}. \quad A = 3 - \frac{9}{5} = \frac{6}{5}$

$$I = \int \frac{6}{5} \frac{1}{u+2} du + \int \frac{9}{5} \cdot \frac{1}{u-3} \cdot du$$

$$= \frac{6}{5} \ln|u+2| + \frac{9}{5} \ln|u-3| + C$$

$$= \frac{6}{5} \ln|e^t + 2| + \frac{9}{5} \ln|e^t - 3| + C$$