1. Practice Problems

(1) **Derive the formula for $E[X]$ for the Bernoulli random variable.**

$$E[X] = \sum_{k \in \text{Range}} k f_X(k) = 1 \cdot p + 0 \cdot (1-p) = p.$$

(2) **Show that $E[X]$ for the Binomial and Hypergeometric random variables can be derived from the previous answer using the properties of Expected Value.**

A binomial random variable Y can be considered as the sum of n independent identically distributed Bernoulli random variables, i.e. $Y = X_1 + \cdots + X_n$, so:

$$E[Y] = E[X_1 + \cdots + X_n] = np.$$

A hypergeomtric random variable Z can be considered as the sum of n *not independent* identically distributed Bernoulli random variables with $p = m/N$, the probability of picking a white marble as the i-th choice: $Z = X_1 + \cdots + X_n$, so:

$$E[Z] = E[X_1 + \cdots + X_n] = \frac{mn}{N}.$$

(3) **Why is Var$[X]$ for the hypergeometric random variable not given by $\frac{mn(N-m)}{N^2}$?**

The suggested value would be correct for the variance if the X_i's were independent; however, the color of each marble is not independent.

(4) **What is a fair price for the following games:**

(a) **A fair coin is flipped until heads comes up, and the player wins a dollar for each tails she flipped before the heads.**

A fair price is the expected value of the game. X = the value of the winnings is a geometric random variable with $p = 1/2$, so $E[X] = 1$. A fair price is one dollar.

(b) **The game runner and the player each roll a six-sided die. If the player has a strictly higher roll, then he wins a dollar.**

Here X = the value of winnings is a Bernoulli random variable with $p = 15/36$, as computed in the midterm. So a fair price for the game is 5/12 dollars, or approximately 42 cents.

(c) **The player watches a road which has red cars pass at a rate of 10 per hour. She wins a dollar for each red car she sees in the hour.**

This time, X is a Poisson random variable with $\lambda = 10$. Then, $E[X] = 10$.

(5) **Suppose I flip a penny, a nickel, a dime, and a quarter. Let X be the value in cents of the coins showing heads. What is $E[X]$ and Var$[X]$?**

The event X can be written as the sum $P + N + D + Q$. Each of these in turn can be written as a function of *independent* Bernoulli trials Y_i by setting $P = Y_1, N = 5Y_2, D = 10Y_3, Q = 25Y_4$. Then,

$$E[X] = E[Y_1 + 5Y_2 + 10Y_3 + 25Y_4] = E[Y_1] + 5E[Y_2] + 10E[Y_3] + 25E[Y_4] = 41/2.$$

$$\text{Var}[X] = \text{Var}[Y_1 + 5Y_2 + 10Y_3 + 25Y_4] =$$
$$\text{Var}[Y_1] + 25\,\text{Var}[Y_2] + 100\,\text{Var}[Y_3] + 625\,\text{Var}[Y_4] = 751/4.$$

Note that we rely on the fact that $Y_1, \ldots, Y_4$ are independent for the variance calculation.