Please submit your answers typed up in TeX, either in hard copy or via e-mail by 1 pm on March 4.

1. **(20 points)** (EC Problem 1.38) Prove that the number of permutations $w \in S_n$ fixed by the fundamental transformation $S_n \xrightarrow{\wedge} S_n$ of Proposition 1.3.1 (reading the standard cycle notation as vector notation) is the Fibonacci number F_{n+1}.

2. **(10 points)** (EC Problem 1.55) If $w = a_1 \cdots a_n \in S_n$ (vector notation) then let $w^r = a_n \cdots a_1$, the reverse of w. Express $\operatorname{inv}(w^r)$, $\operatorname{maj}(w^r)$, and $\operatorname{des}(w^r)$ in terms of $\operatorname{inv}(w)$, $\operatorname{maj}(w)$ and $\operatorname{des}(w)$, respectively. Explain each answer.

3. **(20 points)** (EC Problem 1.69) Let $f(n)$ denote the number of self-conjugate partitions of n all of whose parts are even.

   (a) Express the generating function $\sum f(n)x^n$ as an infinite product. [Hint: Using the hooks as in Figure 1.16 may be helpful.]

   (b) Use the generating function to find the smallest value of n for which $f(n) > 1$. Draw the corresponding partitions.

4. **(20 points)** (From Prof Mark Haiman) A perfect matching on a set S of $2n$ elements is a partition of S into n blocks of two elements each. Perfect matchings form a species M, with $M(S) = \varnothing$ if $|S|$ is odd.

   (a) Express the species M as a sum, product, or composition of two simpler species.

   (b) Use this expression to find the exponential generating series for perfect matchings.

   (c) Deduce that the number of perfect matchings on a set of $2n$ elements is $n!!$, where "double factorial" means the product of all odd positive integers less than $2n$; i.e. $n!! = (2n-1)(2n-3)\cdots 3 \cdot 1$.

   (d) Explain this formula using a counting argument.

5. **(30 points)** Consider the set B of 2×3 matrices whose entries are 0, 1, or 2. The group $G = S_2 \times S_3$ acts naturally on these matrices by permuting rows and columns of the matrix.

   Note that $B = \Phi[D, W]$ where $D = \{(1,1), \ldots, (2,3)\}$ and $W = \{0,1,2\}$. We equip W with a weight in $\mathbb{Q}[x,y]$ given by $w(0) = 1, w(1) = x$, and $w(2) = y$.

   (a) Compute the cycle index polynomial $P_{G:B}(y_1, y_2, \ldots)$.

   (b) Use the Pólya-Redfield Theorem to compute the number of G-reduced matrices in B with j ones, k twos, and $6 - j - k$ zeros.