Why Commutative Algebra?

Based on Eisenbud's Textbook Ch. 1

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Three Great Research Projects

1. Number Theory.
2. Invariant Theory.
3. Algebraic Geometry.

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1. Number Theory.
2. Invariant Theory.
3. Algebraic Geometry.

All required the machinery of commutative algebra to progress.

Number Theory: Fermat's Last Theorem

Theorem (Fermat's Last Theorem)

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Consider $\mathbb{Z}[\zeta]$ where $\zeta = \sqrt[n]{-1}$. $= e^{\pi i/n}$.

$$x^n + y^n = \left(\left(\frac{x}{y}\right)^n + 1\right) y^n$$

$$= \left[\prod_{k=0}^{n} \left(\frac{x}{y} - \zeta^{2k+1}\right)\right] y^n$$

$$= \prod_{k=0}^{n} \left(x - \zeta^{2k+1} y\right) = z^n.$$

Unique Factorization and $\mathbb{Z}[\zeta]$

Many rings of integers do not have unique factorization:

<u>Ex</u> $\mathbb{Z}[\sqrt{-5}]$.

$(1+\sqrt{-5})(1-\sqrt{-5}) = 6 = 2 \cdot 3$

3 irreducible, and not a factor of either term on LHS.

When $n = 23$, $\mathbb{Z}[\zeta]$ does NOT have unique factorization.

How CAN we factor?

Dedekind 1871: Don't factor elements, factor ideals:

$\mathbb{Z}[\sqrt{-5}]$. $(1+\sqrt{-5})(1-\sqrt{-5}) = 2 \cdot 3$.

Prime ideal: set of numbers P, so that $xy \in P \iff x \in P$ or $y \in P$.

$P_2 = (2, 1+\sqrt{-5}) = (2, 1-\sqrt{-5})$

$P_3 = (3, 1+\sqrt{-5})$

$P_3' = (3, 1-\sqrt{-5})$

$(6) = P_2^2 P_3 P_3'$

$P_2^2 = (2)$, $P_3 P_3' = (3)$

$P_2 P_3 = (1+\sqrt{-5})$, $P_2 P_3' = (1-\sqrt{-5})$

If you can factor ideals of a ring into prime ideals uniquely, then the resulting ring is a *Dedekind domain*.

Invariant Theory

Suppose that a group G acts on a space k^n, e.g. $\mathbb{C}^n$.

Which polynomial functions are *invariant* under that action?

Example ($G = \mathbb{Z}_2$)

g = generator. $\curvearrowright \mathbb{C}^1$.

$g \cdot x = -x$.

$g \cdot (p(x)) = p(x)$.

$p(-x) = p(x) \iff p(x) = \mathbb{C}[x^2]$.

S_n and Symmetric Polynomials

$G = S_n$ (symmetric group on n letters)

Let σ act on $(x_1, \ldots, x_n) \in \mathbb{C}^n$ as

$$\sigma . (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

$$e_1 = x_1 + \cdots + x_n$$

$$e_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n$$

$$\vdots$$

$$e_n = x_1 x_2 \cdots x_n .$$

$$(\mathbb{C}[x_1, \ldots, x_n])^{S_n} = \mathbb{C}[e_1, \ldots, e_n]$$

S_n and Symmetric Polynomials

ex $x_1^2 + x_2^2 + \cdots + x_n^2$

$= (x_1 + \cdots + x_n)^2 - 2(x_1x_2 + \cdots + x_{n-1}x_n)$

$= e_1^2 - 2e_2.$

Are Rings of Invariants all Polynomial Rings?

In general, no.

Example ($G = \mathbb{Z}_4$)

Let g act on (x, y) as $g(x, y) = (-y, x)$.

g generator

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Invariants:

$$\begin{aligned} U &= x^2 + y^2 \\ V &= x^2y^2 \\ W &= x^3y - xy^3 \end{aligned}$$

$$\begin{aligned} g \cdot W &= (-y)^3 x - (-y) x^3 \\ &= -y^3 x + y x^3 = x^3 y - x y^3 = W. \end{aligned}$$

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Invariants:

$$\begin{aligned} U &= x^2 + y^2 \\ V &= x^2y^2 \\ W &= x^3y - xy^3 \end{aligned}$$

They satisfy $(U^2 - 4V)V = W^2$

$$((x^2+y^2)^2 - 4x^2y^2)\,x^2y^2 \stackrel{?}{=} (x^3y - xy^3)^2$$

$$(x^4 - 2x^2y^2 + y^4)\,x^2y^2 \qquad \qquad (x^3y - xy^3)^2 = (x^2 - y^2)^2 x^2y^2$$

$$(x^2 - y^2)^2 x^2y^2 = (x^2 - y^2)^2 x^2y^2$$

Are they Finitely-Generated?

Hilbert, in 1890, proved that for many groups, the ring of invariants is finitely generated.

A key part of his proof was the Hilbert basis theorem.

Definition (Noetherian Ring)

Let R be a commutative ring. If every ideal $I \subseteq R$ is finitely generated, then R is called Noetherian.

Theorem (Hilbert Basis Theorem)

If R is Noetherian, then $R[x]$ is Noetherian.

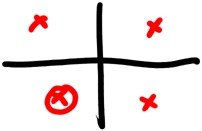
$\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian.

Algebraic Geometry

Theorem (Fundamental Theorem of Algebra, 1806)

A polynomial in $\mathbb{C}[x]$ of degree n has exactly n roots counted with multiplicity.

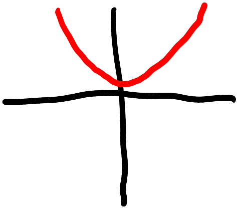
$$p(x) = \sum_{k=0}^{n} a_k x^k \quad \longleftrightarrow$$



$\mathbb{C}$

Example (Ideals in $\mathbb{C}$[x,y])

$$(x^2 - y) \quad \longleftrightarrow$$



$\mathbb{R}^2$

Ideals and Varieties

Definition ($Z(X)$)

Let $X \subseteq k[x_1, \dots, x_n]$ k field.

Define $Z(X) = \{(a_1, \dots, a_n) \mid f(a_1, \dots, a_n) = 0 \ \forall f \in X\}$

Zero-set of X. (Variety)

Definition ($I(S)$)

Let $S \subseteq k^n$, k field.

Define $I(S) = \{f \in k[x_1, \dots, x_n] \mid f(p) = 0 \ \forall p \in S\}$

Ideal of S.

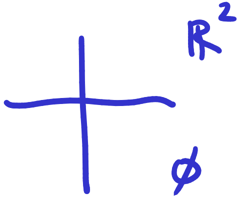
$S \subseteq Z(I(S))$ and $X \subseteq I(Z(X))$

When is this correspondence perfect?

- $\langle x^2+y^2+1\rangle \subseteq \mathbb{R}[x,y]$

$Z(X) = \emptyset$

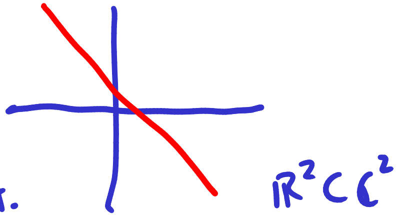
$I(Z(X)) = \mathbb{R}[x,y].$



- $\langle (x+y-1)^k\rangle \subseteq \mathbb{C}[x,y]$

$k=1,2,3,\dots$

all point to the same zero-set.



Hilbert's Nullstellensatz

"zero-places-theorem."

<u>Thm</u> Let k be an algebraically closed field. Then, radical ideals are in bijection with algebraic varieties.

Summary

Mathematical Area	Technology from Commutative Algebra
Number Theory	• UFDs. • Dedekind domain
Invariant Theory	• Symmetric poly's • Hilbert basis theorem.
Algebraic Geometry	• Fund. Thm. of Alg. • Hilbert's Nullst.sz.