Commutative Algebra: Flatness

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Outline

$- \otimes M$, the functor

Let A be a ring, M an A-module.

$$- \otimes_A M : A\text{-mod} \to A\text{-mod}.$$

$$N \to N \otimes_A M. \qquad \text{objects.}$$

$$f : N \longrightarrow P \qquad \text{morphisms.}$$

$$f \otimes_A 1 : N \otimes_A M \longrightarrow P \otimes_A M$$

$$\sum_{i \in I} n_i \otimes m_i \longmapsto \sum_{i \in I} f(n_i) \otimes m_i.$$

$- \otimes M$ is right-exact

$$N' \to N \to N'' \to 0 \quad \text{exact}$$

$$\Big\downarrow \; - \otimes_A M$$

$$N' \otimes M \to N \otimes M \to N'' \otimes M \to 0 \quad \text{exact}$$

Tensoring with M preserves cokernels.

– $\otimes M$ is not left-exact

$$0 \to N' \to N \to N'' \quad \text{exact}$$

$$\Downarrow$$

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \quad \text{NOT necessarily exact}$$

Ex $\quad 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \quad \text{exact}$

$$\Downarrow \; - \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

$$0 \to \boxed{\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

not inj. $\Rightarrow$ not exact.

When is $- \otimes M$ exact?

TFAE conditions characterizing *flatness*.
M is a flat module if

1. $- \otimes M$ is an exact functor.
2. $- \otimes M$ preserves short exact sequences.
3. $f : N' \to N$ injective $\implies$ $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.
4. $f : N' \to N$ injective for N', N finitely generated $\implies$ $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.

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(1) $\Rightarrow$ (2) trivial.

(2) $\Rightarrow$ (1) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\varepsilon} \cdots$

$0 \to A \to B \to \mathrm{im}(\beta) \to 0$

$0 \to \mathrm{coker}(\alpha) \to C \to \mathrm{im}(\gamma) \to 0 \quad \ldots$

(2) $\Rightarrow$ (3) $0 \to N' \to N \to N'' \to 0$

$\otimes M$ preserves exactness.

(3) $\Rightarrow$ (2) preserves kernels & cokernel $\Rightarrow$ exact.

3. $f : N' \to N$ injective $\implies$
   $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.
4. $f : N' \to N$ injective for N', N finitely generated $\implies$
   $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.

(3) $\Rightarrow$ (4) trivial.

(4) $\Rightarrow$ (3). $\sum_{i \in I} x_i \otimes y_i \in \ker(f \otimes 1)$.

$\sum_{i \in I} f(x_i) \otimes y_i = 0 \in N \otimes M$.

Def $N_0' = \langle x_i \rangle$. $N_0 =$ f.g. submodule of N containing $f(N_0')$.

$(f \otimes 1)(\sum x_i \otimes y_i \in N_0') = 0 \Rightarrow \sum x_i \otimes y_i = 0$
$\Rightarrow f \otimes 1$ injective for general modules.

Linear Dependency & Bases

[Approach based on nLab's page on flat modules]

Suppose $v_1, \dots, v_n$ are elements of a k-vector space V. $\exists\, a_1, \dots, a_n \in k$ s.t.

$$\sum_{i=1}^{n} a_i v_i = 0.$$

Take basis $w_1, \dots, w_m$ s.t. $v_i = \sum_{j=1}^{m} b_{ij} w_j$

Then

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_{ij} w_j$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_i b_{ij} \right) w_j . \quad \Rightarrow \quad \sum_{i=1}^{n} a_i b_{ij} = 0 \ \forall j.$$

Translating to Rings

Suppose $x_1, \ldots, x_n \in M$ and $\exists\, a_1, \ldots, a_n \in A$

s.t. $\sum_{i=1}^{n} a_i x_i = 0$

Then, the module M is flat if there exists a collection of elmts $y_1, \ldots, y_m$ s.t.

$x_i = \sum_{j=1}^{m} b_{ij} y_j$ and $\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_{ij} y_j = 0$

$\Rightarrow \sum_{i=1}^{n} a_i b_{ij} = 0$ for all j.

Formal Theorem

A module M is flat if and only if for every finite linear combination

$$\sum_i a_i m_i \equiv 0 \in M, \quad a_i \in A, m_i \in M$$

there are elements n_j and linear combinations

$$m_i = \sum_j b_{ij} n_j \in M, \quad b_{ij} \in A,$$

such that for all j, we have

$$\sum_i a_i b_{ij} \equiv 0 \in R.$$

Defn: Affine Algebraic Variety

[Approach based on Ch 6 of Eisenbud's *Commutative Algebra*]

"Flat families".

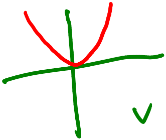
k field. $I \subseteq k[x_1, \dots, x_n]$.

The set of points $V \subseteq k^n$ for which all polynomials in I vanish is called an affine algebraic variety.

The affine coordinate ring $A(V)$ is $k[x_1 \dots x_n]/I$.

<u>Ex</u> $k = \mathbb{R}$. $(y - x^2) \subseteq \mathbb{R}[x,y]$.

$A(V) = \mathbb{R}[x,y]/(y-x^2) \simeq \mathbb{R}[x]$.



Defn: Morphism of Varieties

Let $X \subseteq k^n$, $Y \subseteq k^m$ affine alg vars.

$$f : X \to Y$$
$$(x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_m(x)).$$

$$f^*: A(Y) \to A(X)$$
$$k[y_1, \dots, y_m]/I \to k[x_1, \dots, x_n]/J$$
$$y_i \mapsto f_i(x).$$

What is a family of varieties?

$$\varphi : X \longrightarrow B$$

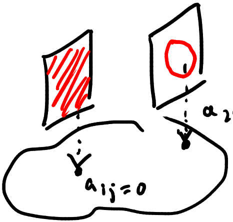
variables & parameters

parameters

$b \in B \rightsquigarrow \varphi^{-1}(b)$ variety in variable-space.

Ex $\underline{a_{20}}x^2 + \underline{a_{11}}xy + \underline{a_{02}}y^2 + \underline{a_{10}}x + \underline{a_{01}}y + \underline{a_{00}}$.

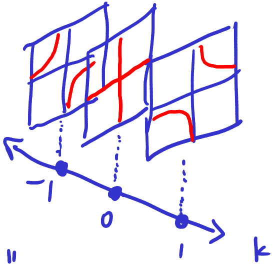
$$\varphi : \underset{(x,y,a)}{k^8} \longrightarrow \underset{(a)}{k^6}$$



Example: Hyperbolas $\{xy - a : a \in k\}$

$$\begin{array}{ccc} k^3 & \longrightarrow & k \\ (x, y, a) & & (a) \end{array}$$

$$X \longrightarrow B$$

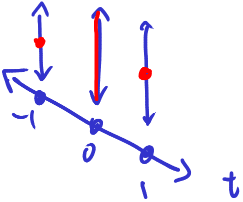


Families that are "nice" have $A(X)$ flat as an $A(B)$-module.

Example: $k[x, t]/(tx)$ not flat as $k[t]$-module

$$k^2 \longrightarrow k$$

$$(t, x) \qquad (t)$$



Indeed,

$M = k[x,t]/(tx)$ is not flat as $k[t]$-module.

$$k[t] \xrightarrow{\times t} k[t] \quad \text{injective}$$

$$k[t] \otimes M \longrightarrow k[t] \otimes M \quad \text{not injective}$$

$$(1 \otimes x) \longmapsto 0.$$