Commutative Algebra: Fractions & Localization (part 2)

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Localization is a functor A-Mod $\to S^{-1}A$-Mod

Context: A ring. $S \subseteq A$ mult subset.

$S^{-1}A$ = localization of A at S.

M is A-module. $S^{-1}M$ is $S^{-1}A$ module.

A-Mod $\qquad\qquad\qquad\qquad$ $S^{-1}A$-Mod

Obj. A-modules $M \longrightarrow S^{-1}M$ Obj $S^{-1}A$-Modules

Morphisms A-mod homom. $\qquad$ Morphisms $S^{-1}A$-mod homomorp.

$f: M \to N \longrightarrow S^{-1}M \to S^{-1}N$.

$f\left(\frac{m}{s}\right) = \frac{1}{s} f(m)$

Localization is an exact functor

Given $N', N, N'' \in A\text{-Mod}$, if

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \text{ exact, then}$$

$$S^{-1}N' \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}N'' \text{ exact.}$$

<u>Proof</u>: WTS: $\mathrm{Ker}(S^{-1}g) = \mathrm{Im}(S^{-1}f)$

$$\boxed{\mathrm{Im}(S^{-1}f) \subseteq \mathrm{ker}(S^{-1}g)} \qquad g \circ f = 0$$

$$\Rightarrow (S^{-1}g) \circ (S^{-1}f)\left(\frac{x}{s}\right) = \frac{(g \circ f)(x)}{s} = 0.$$

Localization is an exact functor

$$\boxed{\operatorname{Ker}(S^{-1}g) \subseteq \operatorname{Im}(S^{-1}f)}$$

Take $\frac{x}{s} \in \operatorname{Ker}(S^{-1}g) \subseteq S^{-1}N.$

$\Rightarrow (S^{-1}g)(\frac{x}{s}) = 0 \Rightarrow \frac{g(x)}{s} = 0 \Rightarrow \exists t \in S$

with $tg(x) = 0. \Rightarrow g(tx) = 0$

$\Rightarrow tx \in \ker(g) \Rightarrow tx \in \operatorname{Im}(f), \; tx = f(y), y \in N'.$

$$S^{-1}f\left(\frac{y}{st}\right) = \frac{f(y)}{st} = \frac{tx}{st} = \frac{x}{s} \Rightarrow S^{-1} \text{ exact } \blacksquare$$

Localization at S is $- \underset{A}{\otimes} S^{-1}A$

$$S^{-1}M = \left\{ \frac{m}{s} : m \in M, s \in S \right\} / \sim$$

Claim: $S^{-1}M \simeq M \otimes_A S^{-1}A$

Construct $f' : M \otimes_A S^{-1}A \to S^{-1}M$.

Then show f' isomorphism.

A-bilinear map from $M \times S^{-1}A \to S^{-1}M$

$$\left(m, \frac{a}{s}\right) \mapsto \frac{am}{s}.$$

$\Rightarrow \exists! f' : M \otimes_A S^{-1}A \to S^{-1}M$ agreeing with f.

Localization at S is $- \otimes S^{-1}A$

f' surjective? $\forall \frac{m}{s} = f'(m, \frac{1}{s})$.

f' injective? Take $x \in M \otimes_A S^{-1}A$.

$$x = \sum_{j=1}^{n} (m_j \otimes \frac{a_j}{s_j}) = \sum_{j=1}^{n} (m_j \otimes \frac{a_j t_j}{s}) \qquad s = \prod s_j$$

$$t_j = \prod_{j \neq i} s_i$$

$$= \sum_{j=1}^{n} (a_j t_j m_j \otimes \frac{1}{s}) = \Big(\underbrace{\sum_{j=1}^{n} a_j t_j m_j}\Big) \otimes \frac{1}{s}.$$

$\Rightarrow x = m \otimes \frac{1}{s}$. Consider $f'(m \otimes \frac{1}{s}) = \frac{m}{s}$.

If $\frac{m}{s} = 0 \Rightarrow \exists t \in S$ such that $tm = 0$.

Localization at S is $- \otimes S^{-1}A$

$$tm = 0 \Rightarrow m \otimes \frac{1}{s} = m \otimes \frac{t}{ts} = tm \otimes \frac{1}{ts}$$

$$= 0 \otimes \frac{1}{ts} = 0$$

$\Rightarrow f'$ injective.

Corollary $S^{-1}A$ is flat as an A-module.

What is a local property?

Let A be a ring. M an A-module.

Recall: Localization at a prime $\mathfrak{p}$, denoted $M_{\mathfrak{p}} = S^{-1}M$ where $S = A \setminus \mathfrak{p}$.

A property P is said to be local if M has property P.

$\iff$ $M_{\mathfrak{p}}$ has property P $\forall \mathfrak{p}$ prime.

Local Property 1: Zero-ness

TFAE: 1) $M = 0$.

2) $M_{\mathfrak{p}} = 0 \quad \forall \mathfrak{p}$ prime.

3) $M_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m}$ maximal.

1) $\Rightarrow$ 2) $\Rightarrow$ 3) (all maximals are prime)

3) $\Rightarrow$ 1) Suppose not. $M \neq 0 \Rightarrow \exists x \in M, x \neq 0$.

$\mathfrak{a} = \operatorname{ann}(x) \subsetneq A$. $\Rightarrow$ $\mathfrak{a} \subseteq \mathfrak{m}$ for some $\mathfrak{m}$ maximal. Consider $\frac{x}{1} \in M_{\mathfrak{m}} = 0 \Rightarrow \frac{x}{1} = 0$.

$\Rightarrow \exists y \in A \setminus \mathfrak{m}$ with $yx = 0$. $\Rightarrow \mathfrak{a} \not\subseteq \mathfrak{m}$ $\Rightarrow\!\Leftarrow$.

Local Property 2: Injectivity

Suppose $\phi: M \to N$. hom of A-modules.

TFAE: 1) ϕ injective.

2) $\phi_{\mathfrak{p}}: M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ inj. $\forall \mathfrak{p}$ prime.

3) $\phi_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ inj. $\forall \mathfrak{m}$ max'l.

Proof: 1) $\Rightarrow$ 2) $0 \to M \to N$ exact

$\Rightarrow$ $0 \to M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ exact. ✓

2) $\Rightarrow$ 3) max'ls are prime.

Local Property 2: Injectivity

3) $\Rightarrow$ 1) $\quad 0 \to \ker(\phi) \to M \xrightarrow{\phi} N$

is exact sequence.

$\Rightarrow 0 \to \ker(\phi)_m \to M_m \to N_m$ exact.

$$\ker(\phi)_m = \ker(\phi_m)$$

ϕ_m inj $\Rightarrow \ker(\phi_m) = 0 \Rightarrow \ker(\phi) = 0$

by local property (1).

Same could be shown for surjectivity.

Local Property 3: Flatness

Let A be a ring, M an A-mod. TFAE:

1) M flat A-module.
2) $M_{\mathfrak{p}}$ flat $A_{\mathfrak{p}}$-module $\forall \mathfrak{p}$ prime.
3) $M_{\mathfrak{m}}$ Flat $A_{\mathfrak{m}}$-module $\forall \mathfrak{m}$ maximal.

Proof: 1) $\Rightarrow$ 2) $M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$

flat $\otimes$ flat = flat. 2) $\Rightarrow$ 3) max $\Rightarrow$ prime.

3) $\Rightarrow$ 1) Let $N \to P$ injective.

$\Rightarrow N_{\mathfrak{m}} \to P_{\mathfrak{m}}$ injective $\Rightarrow M_{\mathfrak{m}} \otimes N_{\mathfrak{m}} \to M_{\mathfrak{m}} \otimes P_{\mathfrak{m}}$

Local Property 3: Flatness

will also be injective.

$$M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}} \cong (M \otimes_A N)_{\mathfrak{m}}$$

$\Rightarrow (M \otimes_A N)_{\mathfrak{m}} \to (M \otimes_A P)_{\mathfrak{m}}$ injective

$\Rightarrow M \otimes_A N \to M \otimes_A P$ inj. (local prop 2)

$\Rightarrow M$ is flat A-module.