Commutative Algebra: Primary Decomposition

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Outline

Motivation: Quadratic Number Fields

$x \in \mathbb{Q}$, unique expression

$$\boxed{x = \frac{p_1^{n_1} \cdots p_r^{n_r}}{q_1^{m_1} \cdots q_s^{m_s}}}$$

p_i, q_j distinct primes.

$\times$ In $\mathbb{Q}(\sqrt{-5})$, $\quad 6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

$\Rightarrow$ (2) is not prime.

$$(6) = (2, 1+\sqrt{-5})^2 \cap (3, 1+\sqrt{-5}) \cap (3, 1-\sqrt{-5})$$

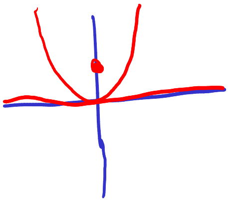
Motivation: Algebraic Varieties

Fix an ideal $I \subseteq \mathbb{C}[x,y]$, e.g.

$$I = \langle xy(y-x^2), y(y-1)(y-x^2) \rangle.$$

The corresponding variety $V(I) \subset \mathbb{C}^2$ is the set of points where all functions in I vanish.

$$I = (y) \cap (y-x^2) \cap (x, y-1)$$



Definition: Primary Ideal

- Let A be a ring.

  $I \subseteq A$ ideal. I is primary if

  $xy \in I \Rightarrow x \in I$ or $y^n \in I$. for some $n > 0$.

- Symmetric version:

  $xy \in I \Rightarrow x \in I$ OR $y \in I$ OR $x^n, y^n \in I$ for some $n > 0$.

- P prime $\Rightarrow A/P$ integral domain.

  I primary $\Rightarrow$ in A/I every zero-divisor is nilpotent.

Definition: Primary Decomposition

Let A be a ring, $I \subseteq A$ ideal.

If I can be written as

$$\underbrace{I = \bigcap_{i=1}^{n} \mathfrak{q}_i}, \quad \mathfrak{q}_i \text{ primary ideal}$$

this is called a primary decomposition.

Such an I is called "decomposable".

Not all ideals are decomposable

Examples tend to be hard to define.

Ideal (0) in the ring of continuous functions on $[0,1]$, $C([0,1])$ is not decomposable.

Def: $\mathfrak{p}$-primary

Prop: $\mathfrak{q}$ primary $\implies$ $\mathfrak{r}(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$.

Claim: $r(\mathfrak{q})$ prime.

$xy \in r(\mathfrak{q}) \Rightarrow \exists n$ s.t. $(xy)^n \in \mathfrak{q} \Leftrightarrow x^n y^n \in \mathfrak{q}$

$\Rightarrow x^n \in \mathfrak{q}$ or $\exists m$ s.t. $y^{nm} \in \mathfrak{q}$

$\Rightarrow x \in r(\mathfrak{q})$ or $y \in r(\mathfrak{q}) \Rightarrow r(\mathfrak{q})$ prime.

Recall: all primes containing $\mathfrak{q}$ contain $r(\mathfrak{q})$. $\blacksquare$

Def: $\mathfrak{p}$-primary

Prop: $\mathfrak{q}$ primary $\implies$ $\mathfrak{r}(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$.

Denote $\mathfrak{p} = \mathfrak{r}(\mathfrak{q})$.

We say that $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal.

Examples: $\mathbb{Z}, \mathbb{Q}[x, y]$

① $\mathbb{Z}$. (27) primary ideal.

$xy \in (27) \Rightarrow x \in (27), y \in (27)$ OR $x, y \in (3)$.

$r(27) = (3)$ (27) is (3)-primary.

② $\mathbb{Q}[x,y]$. $\mathfrak{q} = (x^2, y) \subset \mathfrak{p}^2$.

$x \notin \mathfrak{q}$, but $x \cdot x \in \mathfrak{q}$. $\mathfrak{q}$ is primary

with $r(\mathfrak{q}) = (x, y) = \mathfrak{p}$.

$\mathfrak{q}$ is (x,y)-primary.

Not all prime powers are primary

Let $R = \overset{\text{field}}{k}[x, y, z]/(xy - z^2)$ and $P = (\bar{x}, \bar{z})$.

P prime.

$P^2 = (\bar{x}^2, \bar{x}\bar{z}, \bar{z}^2) = (\bar{x}^2, \bar{x}\bar{z}, \bar{x}\bar{y})$

NOT primary. $\bar{x} \cdot \bar{y} \in P^2$ but $\bar{x} \notin P^2$

and $\bar{y}^n \notin P^2$, for any n.

$P^2 = (\bar{x}) \cap (\bar{x}^2, \bar{y}, \bar{z})$.

Ideal Quotients of Primary Ideals

$(\mathfrak{a} : \mathfrak{b})$ = ideal of elements which multiplied by $\mathfrak{b}$ land in $\mathfrak{a}$.

Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal.

1) $x \in \mathfrak{q}$. $(\mathfrak{q} : x) = (1)$.

2) $x \notin \mathfrak{p}$. $(\mathfrak{q} : x) = \mathfrak{q}$.

3) $x \notin \mathfrak{q}$. $(\mathfrak{q} : x)$ is a $\mathfrak{p}$-primary ideal, i.e. $r(\mathfrak{q} : x) = \mathfrak{p}$.

1st Uniqueness Theorem

① If any $\mathfrak{q}_i, \mathfrak{q}_j$ have the same radical, combine into $\mathfrak{q}_i \cap \mathfrak{q}_j$.

② $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

Let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$. Let $\mathfrak{p}_i = \mathfrak{r}(\mathfrak{q}_i)$. Then

$$\{\mathfrak{p}_i : 1 \leq i \leq n\} = \{\mathfrak{r}(\mathfrak{a} : x) \text{ prime} : x \in A\},$$

hence are independent of the particular decomposition of $\mathfrak{a}$.

Proof

Consider $r(\mathfrak{a}:x)$ prime, $x \in A$.

$$r\left(\left(\bigcap \mathfrak{q}_i\right) : x\right) = r\left(\bigcap_{i=1}^{n} (\mathfrak{q}_i : x)\right)$$

$$= \bigcap_{i=1}^{n} r(\mathfrak{q}_i : x) = \bigcap_{S \subseteq [n]} \mathfrak{p}_i . \qquad r(\mathfrak{q}_i : x) = \begin{cases} (1) \\ \mathfrak{p}_i . \end{cases}$$

$= \mathfrak{p}_i$ for some i.

Take $x_i \notin \mathfrak{q}_i$, $x_i \in \bigcap_{j \neq i} \mathfrak{q}_j$. $r(\mathfrak{a} : x_i) = \mathfrak{p}_i$.

The primary ideals are not unique

$$(x^2, xy) = (x) \cap (x^2, xy, y^2)$$
$$= (x) \cap (x^2, y)$$

distinct primary ideals with radical (x, y).

Def: Associated, Minimal/Isolated, Embedded Primes

Let $\mathfrak{a} \subseteq A$ ideal, $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal primary decomp. and $\mathfrak{p}_i = r(\mathfrak{q}_i)$.

1) $\{\mathfrak{p}_i\}$ = associated primes of $\mathfrak{a}$.

2) Subset minimal under inclusion are the minimal or isolated primes.

3) The complement of minimal primes are embedded primes.

Associated Primes of (0)

Suppose that (0) is decomposable.

Then, the associated primes of (0) constitute the set of zerodivisors.

$$D = \bigcup_{x \neq 0} (0 : x) = \bigcup_{x \neq 0} r(0 : x)$$

$$= \bigcup \text{assoc. primes of zero.}$$

zero divisors

$$\mathfrak{N} = \bigcap \text{minimal primes of zero.}$$

nilpotents.

Localization & Primary Ideals

Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal,

$S \subseteq A$ be a mult. subset.

1) $S \cap \mathfrak{p} \neq \emptyset$. $S^{-1}\mathfrak{q} = S^{-1}A$.

$s \in S \cap \mathfrak{p} \Rightarrow s^n \in S \cap \mathfrak{q} \Rightarrow \frac{s^n}{1} \in S^{-1}\mathfrak{q} = (1)$.

2) $S \cap \mathfrak{p} = \emptyset$. $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.

with contraction (pre-image) $\mathfrak{q}$.

2nd Uniqueness Theorem

Any embedded prime comes with minimal primes it contains.

Let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$, and let $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_m}\}$ be an isolated set of prime ideals of $\mathfrak{a}$. Then $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_m}$ is independent of the decomposition.

In particular, the isolated primary components are uniquely determined by $\mathfrak{a}$.

embedded primary components are not uniquely determined.