Commutative Algebra: Integral Extensions

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Outline

- Definitions
- Examples: Extensions of $\mathbb{Z}$ & Maps to the Line
- Basic Results

Def: Integral Element

Let $A \subset B$ be rings.

$x \in B$ is integral over A if it satisfies a monic polynomial relation:

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

with $a_i \in A$.

Integral: Equivalent conditions (Prop 5.1)

1) x is integral over A.
2) $A[x]$ is a finitely-generated A-module.
3) There exists a subring $C \subseteq B$ s.t. $A[x] \subset C$, and C is a f.g. A-module.
4) There exists a faithful $A[x]$-module M which is finitely-gen. as an A-module

Recall: M faithful $\Leftrightarrow$ $\mathrm{Ann}(M) = 0$.

Proof

1) $\Rightarrow$ 2) x integral $\Rightarrow x^n + a_1 x^{n-1} + \cdots + a_n = 0$.

$\Rightarrow x^n = -a_1 x^{n-1} - \cdots - a_n \Rightarrow A[x] \simeq \bigoplus_{k=0}^{n-1} A x^k$.

2) $\Rightarrow$ 3) $C = A[x]$ f.g. A-module.

3) $\Rightarrow$ 4) Let $M = C$. f.g. A-module by hyp,

faithful because $1 \in C$. $yC = 0$

$\Rightarrow y \cdot 1 = 0 \Rightarrow y = 0$.

4) $\Rightarrow$ 1) Apply Cayley-Hamilton Thm.

$xM \subset M$. $\Rightarrow \phi_x \in \mathrm{End}(M)$. $\Rightarrow \phi_x^n + a_1 \phi_x^{n-1} + \cdots + a_n = 0$

$(x^n + a_1 x^{n-1} + \cdots + a_n) M = 0 \Rightarrow x$ integral.

Def: Integral Closure

Let $A \subseteq B$.

Let $C = \{x \in B : x \text{ integral over } A\}$.

Then, C is a subring of B, called
prop
the integral closure of A in B.

Proof: $x, y \in C$. $A[x]$ f.g. A-module.

y integral over $A \Rightarrow$ int over $A[x]$

$\Rightarrow A[x][y]$ f.g. $A[x]$-module. $\Rightarrow$ f.g. A-module.

Def: Integral Closure

If integral closure of A in B is equal to A, then A is called "integrally closed in B".

If B not specified, A is integrally closed in $\mathrm{Frac}(A)$.

Def: Integral Homomorphism, Algebra

If $f : A \to B$ ring hom.

Then, f integral homomorphism, if B is an integral extension of $f(A)$.

B is an integral $f(A)$-algebra or A-algebra.

Integral, not Int Closed: $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{5}]$

$\sqrt{5}$ is integral over $\mathbb{Z}$ since

$x^2 - 5 = 0.$

$\mathbb{Z}[\sqrt{5}]$ not int closed in $\mathbb{Q}[\sqrt{5}]$

because $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$ satisfies

$x^2 - x - 1 = 0$ $\quad x = \frac{1 \pm \sqrt{5}}{2}$

Integral, Int Closed Extension: $\mathbb{Z} \subseteq \mathbb{Z}[i]$

Gaussian integers $i = \sqrt{-1}$.

Integral: i satisfies $x^2 + 1 = 0$.

Integrally closed: consequence of $\mathbb{Z}[i]$ being UFD.

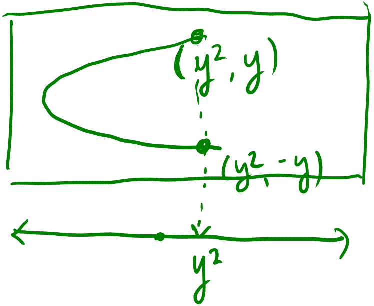
Non-Integral Extension: $\mathbb{Z} \subseteq \mathbb{Q}$

e.g. $\frac{1}{2}$ is not integral over $\mathbb{Z}$.

$\frac{1}{n}$ as well.

Given $\frac{1}{n}$, $\mathbb{Z}\left[\frac{1}{n}\right]$ is not finitely-generated as a $\mathbb{Z}$-module.

Integral: Parabola to Line



$V \subseteq \mathbb{A}^2 \qquad k[x,y]/(y^2-x)$

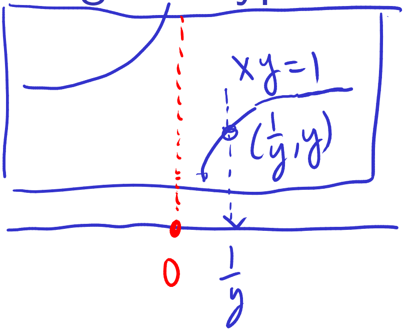
$\cup$

$\mathbb{A}^1 \qquad k[x].$

integral extension $k[x] \subset k[x,y]/(y^2-x)$.

integral dependence $\Rightarrow$ map is surjective with finite fibers.

Non-Integral: Hyperbola to Line



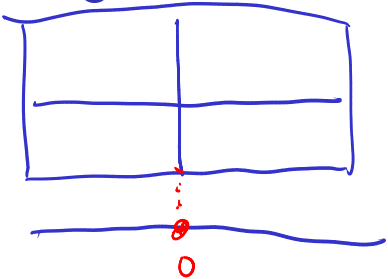
$$\mathbb{A}^2 \qquad k[x,y]/(xy-1)$$
$$\cup$$
$$\mathbb{A}^1 \qquad k[x]$$

y is not integral over $k[x]$

$xy-1=0$ is not monic.

$x=0$: $0 \cdot y - 1 = 0 \Rightarrow$ no solutions for y.

Non-Integral: Coordinate Axes to Line



$V \subseteq \mathbb{A}^2 \quad k[x,y]/(xy)$

$\mathbb{A}^1 \quad k[x]$

$k[x] \subseteq k[x,y]/(xy)$ not integral.

y satisfies $xy = 0 \leftarrow$ not monic.

$x=0$: $0 \cdot y = 0$ has infinite number of solutions.

F.G. Algebra + Integral $\Longrightarrow$ F.G. Module

Suppose $B = A[x_1, \ldots, x_n]$ where x_i integral for all i, then B is a finitely-generated A-module.

Proof: $B = A[x_1]$ f.g A-module.

Assume that $A_{n-1} = A[x_1, \ldots, x_{n-1}]$ is f.g. A-module. Then $A_{n-1}[x_n]$ is f.g. A_{n-1} module. $\Rightarrow$ $A_{n-1}[x_n]$ f.g. A-module. $\square$

Integral dependence is transitive

If $A \subset B$ is integral and $B \subset C$ integral, then $A \subset C$ is also integral.

Proof: $x \in C$ integral over B, then we have $x^n + b_1 x^{n-1} + \dots + b_n = 0$. $\Rightarrow B' = A[b_1, \dots, b_n]$

x integral over B', B' f.g. A-module.

$B'[x]$ f.g. B'-module, B' f.g. A-module

$\Rightarrow B'[x]$ f.g. A-module $\Rightarrow x$ integral / A.

Quotient preserves integral dependence

Let $A \subseteq B$ be an integral extension.
Let $\mathfrak{b} \subseteq B$ be an ideal with
$\mathfrak{a} = A \cap \mathfrak{b}$ ideal. Then $B/\mathfrak{b}$ is int
over $A/\mathfrak{a}$.

Proof: $\forall x \in B$, $\exists$ monic polynomial

$x^n + a_1 x^{n-1} + \cdots + a_n = 0$ mod out by $\mathfrak{b}$.

$\bar{x}^n + \bar{a}_1 \bar{x}^{n-1} + \cdots + \bar{a}_n = 0$ $\bar{x} \in B/\mathfrak{b}$, $a_i \in A/\mathfrak{a}$.

Localization preserves integral dependence

Let $A \subseteq B$ integral extension.

$S \subseteq A$ mult. closed subset.

Then $S^{-1}A \subseteq S^{-1}B$ is integral.

Proof Take $\frac{x}{s} \in S^{-1}B$. Then

$$\frac{1}{s^n}\left(x^n + a_1 x^{n-1} + \cdots + a_0 = 0\right)$$

$$\left(\frac{x}{s}\right)^n + \left(\frac{a_1}{s}\right)\left(\frac{x}{s}\right)^{n-1} + \cdots + \frac{a_0}{s^n} = 0. \Rightarrow \frac{x}{s} \text{ int.}$$

over $S^{-1}A$.