Commutative Algebra: Chain Conditions

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Outline

a.c.c. & Maximal Condition

① ascending chain condition (a.c.c.)

Let Σ be a poset with $\leq$.

Σ satisfies a.c.c if for any sequence $x_1 \leq x_2 \leq x_3 \leq \cdots$, the sequence is eventually stationary ($\exists n$ s.t. $x_n = x_{n+1} = \cdots$)

② Maximal condition:

$X \subseteq \Sigma$ nonempty subset has a maximal element.

a.c.c. & Maximal Condition

a.c.c $\Rightarrow$ maximal. (We prove contrap.)

not maximal $\Rightarrow$ not. a.c.c.

Inductively build $x_1 \leq x_2 \leq x_3 \leq \dots$

Since this is not stationary, acc fails.

maximal $\Rightarrow$ a.c.c.

Take set $(x_i)_{i=1,2,\dots}$ the maximal

element $= x_n$. Since $x_n \not\geq x_k$ $k > n$,

$x_n \leq x_k$ by setup $\Rightarrow x_n = x_k$ $\forall k > n$.

d.c.c. & Minimal Condition

descending chain condition (d.c.c)

$(\Sigma, \leq)$ poset. $x_1 \geq x_2 \geq x_3 \geq \cdots$ is stationary, i.e. $\exists n$ s.t. $x_n = x_{n+1} = \cdots$.

minimal condition:

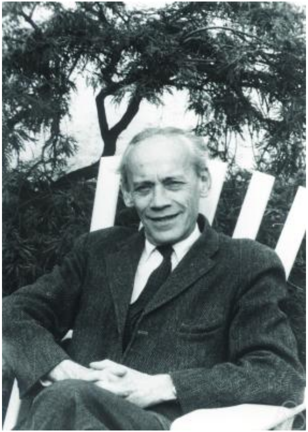
$X \subset \Sigma$ nonempty subset has a minimal element.

Noether & Artin

Σ = submodules of M

$\leq \; = \; \subseteq$

Noether(ian)



Artin(ian)

$(\Sigma, \leq)$ a.c.c. $\Rightarrow$ M Noetherian

d.c.c. $\Rightarrow$ M Artinian

Examples: a.c.c. & d.c.c.

Modules Noetherian & Artinian.

① Finite abelian group e.g. $\mathbb{Z}/8\mathbb{Z}$
($\mathbb{Z}$-module).

Only finitely many submodules!

② Field k, e.g. $\mathbb{Q}$.

Only $\mathbb{Q}$-submodules are (0), $\mathbb{Q}$.

$\Rightarrow$ chain conditions easily satisfied.

Examples: a.c.c., but not d.c.c.

Noetherian Modules, not Artinian.

① $\mathbb{Z}$. $(2) \supset (4) \supset (8) \supset \cdots \supset (2^n) \supset \cdots$

Given $(n) = (p_1^{r_1} \cdots p_k^{r_k}) \subseteq \cdots$ must stabilize.

② $k[x]$, k field.

Descending: $(x) \supset (x^2) \supset \cdots \supset (x^n) \supset \cdots$ not stationary

Ascending: $(f) = (f_1^{k_1} \cdots f_r^{k_r})$ f_i irreducible

Examples: d.c.c. but not a.c.c.

Artinian module, not Noetherian.

$G = \mathbb{Q}/\mathbb{Z}$. $H < G$, $H = \{ \frac{n}{2^k} : k \geq 0 \}$.

H_0 = elements of order $2^0 = 1$ $\{0\}$

H_1 = " " " $2^1 = 2$ $\{0, \frac{1}{2}\}$

H_2 = same, $2^2 = 4$, $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$.

$H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ Ascending not stationary.

Descending: $H_n \supseteq \ldots$ must stabilize.

Examples: not a.c.c. or d.c.c.

Not Noetherian or Artinian.

(1) $k[x_1, x_2, x_3, \dots]$, k field.

asc: $(x_1) \subset (x_1, x_2) \subset \dots \subset (x_1, \dots, x_n) \subset \dots$

desc: $(x_1) \supset (x_1^2) \supset (x_1^3) \supset \dots \supset (x_1^n) \supset \dots$

(2) $\overline{\mathbb{Z}}$, int. closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$.

asc: $(\sqrt{2}) \subsetneq (\sqrt[4]{2}) \subsetneq \dots \subsetneq (\sqrt[2^k]{2}) \subsetneq \dots$

des: $(2) \supsetneq (4) \supsetneq (8) \supsetneq \dots \supsetneq (2^k) \supsetneq \dots$

a.c.c. & f.g. submodules

Thm Let M be a module.
M is Noetherian $\iff$ every $N \subset M$ submodule is f.g.

Proof ($\Rightarrow$) Contrapositive. Suppose $\exists N \subset M$ not f.g. Take $x_1, x_2, \ldots$ elmts of N s.t. $x_i \notin (x_1, \ldots, x_{i-1})$. Then $(x_1) \subset (x_1, x_2) \subset \cdots$ is not stationary $\Rightarrow$ M not Noetherian.

a.c.c. & f.g. submodules

($\Leftarrow$) consider $N_1 \subset N_2 \subset N_3 \subset \cdots$ Let

$N = \bigcup_i N_i$ $\leftarrow$ submodule $\Rightarrow$ $N = (x_1, \ldots, x_m)$

Each $x_i \in N_i$ for some i. Take the

highest index k for each of the m elmts

then, $N \subset N_k \Rightarrow N = N_k \Rightarrow N_{k+1} = N_k, \ldots$

$\Rightarrow$ Chain is stationary.

Exact sequences

Thm Let $0 \to \underline{\underline{M'}} \xrightarrow{i} \boxed{M} \xrightarrow{\pi} \underline{\underline{M''}} \to 0$ be an exact sequence of modules.

Then: (1) M Noetherian $\Leftrightarrow$ M', M'' Noeth.

(2) M Artinian $\Leftrightarrow$ M', M'' Artinian.

Proof: Artinian case

M Artinian $\Leftrightarrow$ M', M'' Artinian.

($\Rightarrow$) Let $N_1 \supset N_2 \supset \dots$ be a chain in M'

these map to a chain in M. M Artinian

$\Rightarrow i(N_k) = i(N_n)$ for all $k > n$. $\Rightarrow N_k = N_n$ for

all $k > n$. $\Rightarrow$ d.c.c. holds for M'.

Consider a chain $N_1 \supset N_2 \supset \dots$ in M''.

$\pi^{-1}(N_1) \supset \pi^{-1}(N_2) \supset \dots$ is a chain in M.

$\Rightarrow \pi^{-1}(N_k) = \pi^{-1}(N_n)$ for all $k > n$.

Proof: Artinian case

Since π surjective, $\pi^{-1}(N_n) = \pi^{-1}(N_k)$

$\Rightarrow N_n = N_k$. Why? If $x \in N_k$, $x \notin N_n$ then

$\pi^{-1}(x) \in \pi^{-1}(N_k) \Rightarrow \pi^{-1}(x) \in \pi^{-1}(N_n)$. $\rightarrow x \in N_n$.

($\Leftarrow$) Take a chain $N_1 \supset N_2 \supset \cdots$ in M.

$i^{-1}(N_1) \supset i^{-1}(N_2) \supset \cdots$ stationary $\Rightarrow i^{-1}(N_k) = i^{-1}(N_n)$

for all $k > n$. $\pi(N_1) \supset \pi(N_2) \supset \cdots$ stationary

$\Rightarrow \pi(N_k) = \pi(N_n)$ $k > n$. Suppose $N_{n+1} \neq N_n$,

$\exists\, x \in N_n$ s.t. $x \notin N_{n+1}$. $\pi(x) \in \pi(N_{n+1}) = \pi(x_{n+1})$

Proof: Artinian case

$\pi(x) = \pi(x_{n+1}) \Rightarrow \pi(x - x_{n+1}) = 0.$

$x - x_{n+1} \notin N_{n+1}. \Rightarrow \exists\, y \in M'$ s.t. $i(y) = x - x_{n+1}.$

$x - x_{n+1} \in N_n \Rightarrow y \in i^{-1}(N_n) \Rightarrow y \in i^{-1}(N_{n+1}). \Rightarrow \Leftarrow.$

$\blacksquare$

Cor: Direct Sums

Let $M_1, \ldots, M_n$ be Noetherian modules, then $\bigoplus_{i=1}^{n} M_i$ is also Noetherian. Same for Artinian.

Def: Noetherian / Artinian Rings

If R is a Noetherian (Artinian) module when considered as a module over itself, then R is a Noetherian (Artinian) ring.

Equiv, ideals of R satisfy the a.c.c. (or d.c.c).

F.g. A-Modules

If A is a Noetherian (Artinian) ring, M is a f.g. A-module, then M is Noetherian (Artinian).

Quotients

Let A be a Noetherian / Artinian ring. Then so is $A/\mathfrak{p}$, $\mathfrak{p}$ ideal.

Proof: Ideals of $A/\mathfrak{p}$ are in bijection with ideals of A containing $\mathfrak{p}$.

Def: Composition Series

Let M be a module.

Take $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$.

Such that this chain is maximal (no other submodules can be inserted to lengthen the chain).

This is a "composition series" of M

Comp Series have Fixed Length

i.e. given $M_0 \subset \cdots \subset M_n$ a composition series, every other composition series has length n. $\leftarrow$ length(M).

Idea of Proof Take any proper submodule $N \subset M$. $M_i \cap N$ gives a composition series of N (after discarding repeats) $\Rightarrow$ smaller length than M.

Comp Series iff Chain Conditions

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M.$$

$\Leftrightarrow$ M Noetherian and Artinian.

$(\Rightarrow)$. $M_1' \subset M_2' \subset \cdots \subset$ throwing away repeats gives a composition series for the highest unique term — this has bounded length.

$(\Leftarrow)$ $M = M_0$. Take M_1 to be maximal proper submodule of M. Continue inductively.

Comp Series iff Chain Conditions

Gives a chain

$M = M_0 \supset M_1 \supset M_2 \supset \cdots$ maximal

By d.c.c., this stabilizes.

That gives a composition series.

Vector Spaces & Chain Conditions

Let k be a field. V a k-vector space.

TFAE: (1) V has finite dim.

(2) V has finite length.

(3) V satisfies a.c.c.

(4) V satisfies d.c.c.

Proof (1) $\Leftarrow$ (2). Contrapositive. Suppose V has infinite dim.

$0 \subsetneq (e_1) \subsetneq (e_1, e_2) \subsetneq \cdots$

$V \supsetneq V/(e_1) \supsetneq V/(e_1, e_2) \supsetneq \cdots$

$$(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$$

Suppose A is a ring s.t. $\mathfrak{m}_i \subset A$ maximal, $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Then A is Noetherian iff A is Artinian.

<u>Proof:</u> $A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \supset (0)$

This is a maximal chain of ideals.

$A/\mathfrak{m}_1$, $\mathfrak{m}_1/\mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_1(A/\mathfrak{m}_2)$, etc. are all vectorspaces over fields.

$$(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$$

Each vector space is Artinian iff Noetherian.

$\Rightarrow$ Ring constructed from those vector spaces is also Noetherian iff Artinian (use exact sequences inductively.)