Commutative Algebra: Noetherian Rings

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Outline

Operations Preserving Noetherian Condition

Hilbert Basis Theorem

Noetherian $\implies$ All Ideals have Primary Decomposition

Recall– Def: Noetherian

① ascending chain condition on ideals:

$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ stationary.

② Maximal condition on ideals:

Any collection of ideals has a maximal element.

③ All ideals are finitely generated.

These equiv. conditions characterize a Noetherian ring.

Homomorphisms

Let A be a Noetherian ring.
$\phi : A \to B$ surj. homomorphism of rings. Then B Noetherian.

Proof $B \simeq A/\mathfrak{a}$. Ideals of B are in order-preserving bijection with ideals of A containing $\mathfrak{a}$
$\Rightarrow$ since A satisfies maximal condition, so does B.

Finitely-Generated Modules

Let $A \subseteq B$, A Noetherian, B finitely-generated as an A-module. Then B is a Noetherian ring.

Proof: B f.g. A-module $\Rightarrow$ Noetherian as an A-module.

Any A-submodule of B is finitely generated. Since every B-submodule is an A-submodule, these are f.g.

$\Rightarrow$ B Noetherian.

Localization

Let A be a Noetherian ring. $S \subseteq A$ mult. subset. Then $S^{-1}A$ is also Noetherian.

<u>Proof</u>: Ideals of $S^{-1}A$ are in order-preserving bijection with the ideals of A not meeting S. A satisfies maximal cond $\Rightarrow S^{-1}A$ has maximal condition.

Localization

Alternatively,

$$\begin{array}{ccc} \mathfrak{a} \subseteq A & \longrightarrow & S^{-1}\mathfrak{a} \subseteq S^{-1}A \\ \parallel & & \parallel \\ (x_1, \ldots, x_n) & & \left(\frac{x_1}{1}, \ldots, \frac{x_n}{1}\right) \end{array}$$

A has fin gen ideals $\Rightarrow$ $S^{-1}A$ has fin gen ideals.

David Hilbert

Hilbert Basis Theorem

Let A be a Noetherian ring.
Then, $A[x]$ is also Noetherian.

Proof

Take $\mathfrak{a} \subseteq A[x]$. WTS: $\mathfrak{a}$ is finitely-generated.

Define $\mathrm{in}(f)$ = coefficient of the highest power of x in f.

$\mathrm{in}\,\mathfrak{a} = \{\mathrm{in}(f) : f \in \mathfrak{a}\}$ ideal in A.

$b, c \in \mathrm{in}\,\mathfrak{a} \Rightarrow f_b = bx^r + \cdots,\ f_c = cx^s + \cdots \in \mathfrak{a}$.

$r \geq s$. $f_b + x^{r-s} f_c \in \mathfrak{a} \Rightarrow b + c \in \mathfrak{a}$.

Proof

$\mathrm{in}\,\mathfrak{a} \subset A$ ideal $\Rightarrow \mathrm{in}\,\mathfrak{a}$ f.g.

$\Rightarrow \mathrm{in}\,\mathfrak{a} = \langle a_1, \dots, a_n \rangle$.

$\Rightarrow \exists f_1, \dots, f_n$ s.t. $f_i = a_i x^{r_i} +$ lower order terms.

finitely many polynomials $\Rightarrow \max\{r_i\} = m$ will be finite.

Given $g \in \mathfrak{a}$, $g = f + \tilde{g}$, where

$f \in \langle f_1, \dots, f_m \rangle$, $\deg(\tilde{g}) < m$.

Proof

If $\deg(g) < m$, we are done.

Suppose $\deg(g) \geq m$. Then,

$g = ax^k +$ lower-order terms, $k \geq m$

$a = \sum_{i=1}^{n} c_i a_i$, $c_i \in A$

$f = \sum_{i=1}^{n} c_i x^{k-r_i} f_i$ has leading term ax^k.

$g = f + \tilde{g}$ $\quad \deg(\tilde{g}) < \deg(g)$

Repeating inductively, $\tilde{g}$ will have $\deg < m$.

Proof

$$\mathfrak{a} = \langle f_1, \dots, f_n \rangle + \mathfrak{a} \cap M$$

$$M = \langle 1, x, x^2, \dots, x^{m-1} \rangle \quad A\text{-module}$$

$\mathfrak{a} \cap M$ f.g. A-module $\Rightarrow$ Noetherian as an A-module $\Rightarrow$ Noetherian as $A[x]$-mod.

$\Rightarrow$ f.g. by $(g_1, \dots, g_k)$

$\mathfrak{a} \subset A[x]$ is generated by (f_i, g_j)

$\Rightarrow \mathfrak{a}$ f.g. $\Rightarrow A[x]$ Noetherian $\blacksquare$

Corollary: F.g. Algebras over Noetherian rings

$A[x]$ Noetherian.

Induction: $A[x_1, \ldots, x_n]$ Noetherian.

$\Rightarrow$ $A[x_1, \ldots, x_n]/I$ Noetherian.

Lemma: Irreducible Decomposition

Def: ideal $\mathfrak{a} \subsetneq A$ is irreducible if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \Rightarrow \mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}.$$

Lemma Let A be Noetherian. Every ideal $\mathfrak{a} \subset A$ can be written as an intersection of finitely many irred. ideals. (irreducible decomposition).

Lemma: Irreducible Decomposition

Proof: Suppose not. A Noetherian

$\Rightarrow$ the set of ideals without irred decomp

must have a maximal element.

Consider $\mathfrak{a}$ of this type.

$\mathfrak{a}$ not irreducible $\Rightarrow$ $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with

$\mathfrak{b} \supsetneq \mathfrak{a}$, $\mathfrak{c} \supsetneq \mathfrak{a}$. $\mathfrak{a}$ maximal $\Rightarrow$ $\mathfrak{b}$ and

$\mathfrak{c}$ have irred decomp. $\Rightarrow$ $\mathfrak{a}$ has

irred decomp. $\Rightarrow\Leftarrow$.

Lemma: Irreducibles are Primary

Proof Take $\mathfrak{a} \subseteq A$ irreducible. A Noetherian.

$\mathfrak{a} \subseteq A$ primary $\Longleftrightarrow$ $(0) \subseteq A/\mathfrak{a}$ primary.

WTS: (0) irreducible $\Rightarrow$ (0) primary.

$xy \in (0) \rightsquigarrow$ show $y = 0$ or $x^n = 0$ for some $n > 0$.

Assume $y \neq 0$. Then consider

$\mathrm{ann}(x) \subseteq \mathrm{ann}(x^2) \subseteq \dots$ ascending chain.

$\exists n$ s.t. $\mathrm{ann}(x^n) = \mathrm{ann}(x^{n+1}) = \dots$

Lemma: Irreducibles are Primary

<u>Claim</u>: $(x^n) \cap (y) = (0)$.

Suppose not. Then $a \neq 0$, $a \in (x^n)$

and $a \in (y)$. $a = by \Rightarrow ax = 0$.

$a \in (x^n) \Rightarrow \boxed{a = cx^n} \Rightarrow cx^{n+1} = 0$.

$\Rightarrow c \in \mathrm{ann}(x^{n+1}) \Rightarrow c \in \mathrm{ann}(x^n). \Rightarrow a = 0$.

(0) irreducible $\Rightarrow (x^n) = 0$

$\Rightarrow (0)$ primary. $\blacksquare$

Theorem

Let A be a Noetherian ring.
Any ideal $\mathfrak{a} \subseteq A$ has a
primary decomposition.