Commutative Algebra: Ideals

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Theme: Ideals & Quotients

Special properties of ideal

$$\Longleftrightarrow$$

special properties of the quotient ring

Definition: Prime Ideal

Let $P \subseteq A$ be an ideal $\neq A$.

For any $z \in P$, if $z = xy$, then either $x \in P$ or $y \in P$.

P satisfying this property is a prime ideal.

Quotient: Integral Domain

$$xy \in P \Rightarrow x \in P \text{ or } y \in P$$

quotient

$$xy \equiv 0 \in A/P \Rightarrow x \equiv 0 \text{ or } y \equiv 0$$

in A/P.

$\Rightarrow A/P$ has no zerodivisors

$\Rightarrow A/P$ is an integral domain.

Examples

1) irreducible polynomials in poly rings over a field

$(x-1) \subseteq \mathbb{C}[x]$, $(x^2+1) \subseteq \mathbb{R}[x]$

2) In $\mathbb{Z}$, the prime ideals are $p\mathbb{Z}$, p prime. $(2, 3, 5, 7, 11, \ldots)$

Non-examples: $(x^2+1) \subseteq \mathbb{C}[x]$

$6\mathbb{Z} \subseteq \mathbb{Z}$ $2 \cdot 3 = 6 \in 6\mathbb{Z}$, but not $2, 3$.

Preimages of Prime Ideals are Prime

Let $f : A \to B$ ring hom.

If $P \subseteq B$ is a prime ideal,

then $f^{-1}(P)$ is also a prime ideal.

1) $f^{-1}(P)$ ideal.

$a, b \in f^{-1}(P)$, $f(a+b) = f(a) + f(b)$

$\in P$, since P closed under addition.

$a \in A$, $x \in f^{-1}(P)$. $f(ax) = f(a)f(x) \in P$.

Preimages of Prime Ideals are Prime

2) $xy \in f^{-1}(P)$.

$f(xy) = f(x)f(y) \in P$

Since P prime, $f(x) \in P$ or $f(y) \in P$

$\Rightarrow x \in f^{-1}(P)$ or $y \in f^{-1}(P)$,

$\Rightarrow f^{-1}(P)$ prime.

Images of Prime Ideals not Generally Prime

<u>ex</u> $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

$(2) \dashrightarrow (2)$ not an ideal.

<u>ex</u> $\mathbb{Q}[y] \longrightarrow \mathbb{Q}[x]$

$y \longmapsto x^2-1$

$(y) \dashrightarrow (x^2-1)$ not prime.

Definition: Maximal Ideal

Let $\mathfrak{m} \subseteq A$ be an ideal s.t. for all ideals I satisfying $\mathfrak{m} \subseteq I \subsetneq A$, either $\mathfrak{m} = I$ or $A = I$.

Quotient: Field

$\mathfrak{m} \subsetneq A$ maximal

$A/\mathfrak{m}$ = field.

* Only ideals of a field F are (0) and F.
* Bijection between ideals of A containing $\mathfrak{m}$ and ideals of $A/\mathfrak{m}$.

Maximal Ideals are Prime

Ideal $P \subseteq A$ prime $\Leftrightarrow$ A/P integral domain

$\mathfrak{m} \subseteq A$ maximal $\Leftrightarrow$ $A/\mathfrak{m}$ field

$\Rightarrow$ $A/\mathfrak{m}$ int domain

$\Rightarrow$ $\mathfrak{m}$ prime.

A Maximal Ideal Exists in Every Ring

nonzero

<u>Zorn's Lemma</u> Let Σ be a non-empty family such that every chain in Σ has an upper bound. Then, Σ has a maximal element.

- $\Sigma_1 = \{\text{ideals of } A\}$.
- Σ_1 non-empty, b/c $(0) \in \Sigma_1$.

A Maximal Ideal Exists in Every Ring

- Given a chain of ideals

  $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$

  there is an upper bound $\bigcup_{i \in I} \mathfrak{a}_i = \mathfrak{a}$.

  Suppose $\mathfrak{a} = A$, then $1 \in \mathfrak{a}$, then $\exists i$ s.t. $1 \in \mathfrak{a}_i$. $\Rightarrow \Leftarrow$ $\mathfrak{a}$ proper.

$\Rightarrow$ Zorn: there is a maximal ideal.

Definition: Principal Ideal Domain

A *principal ideal domain* (PID) is an integral domain in which every ideal is principal.

Claim

Every non-zero prime ideal in a PID is maximal.

$P = (x) \neq (0)$. Suppose $(x) \subsetneq (y)$. $\Rightarrow x = yz$.

$yz \in (x)$ prime $\Rightarrow$ $\cancel{y \in (x)}$ or $z \in (x)$

$\Rightarrow z = xt \Rightarrow x = yxt = (yt)x$

$\Rightarrow yt = 1 \Rightarrow (y) = (1)$. $\Rightarrow (x)$ max'l.

Definition: Nilradical $\mathfrak{N}$

The nilradical $\mathfrak{N}$ is the set of nilpotent elements of the ideal.

Claim

The nilradical is an ideal.

$x \in \mathfrak{N} \subseteq A$, $a \in A$. $x \in \mathfrak{N} \Rightarrow \exists k, x^k = 0.$

$(ax)^k = a^k x^k = 0. \Rightarrow ax \in \mathfrak{N}$ $\rightsquigarrow$ closed under mult by A

$x, y \in \mathfrak{N} \Rightarrow \exists m, n$ s.t. $x^m = 0$, $y^n = 0$.

$(x+y)^{m+n-1} = \sum_{k=0}^{m+n-1} a_k x^k y^{m+n-k-1} = 0.$

$\rightsquigarrow$ closed under addition.

Quotient: Reduced Ring

$\mathfrak{N} \subseteq A$ nilradical.

$A/\mathfrak{N}$ has no nonzero nilpotents $\Rightarrow$ reduced.

Suppose $x \in A/\mathfrak{N}$ is nilpotent, then

$x^n = 0 \Rightarrow \exists\, \tilde{x} \in A$ s.t. $\tilde{x}^n \in \mathfrak{N}$

$\Rightarrow \exists k$ s.t. $(\tilde{x}^n)^k = 0 \Rightarrow \tilde{x} \in \mathfrak{N}$.

$\Rightarrow x = 0$.

$\mathfrak{N} =$ intersection of all primes $=: \tilde{\mathfrak{N}}$

$(\subseteq)$ $f \in \mathfrak{N}$. $f^n = 0 \in$ all P prime.

$\Rightarrow f \in$ all P prime $\Rightarrow f \in \tilde{\mathfrak{N}}$.

$(\supseteq)$ Take f s.t. $f^n \neq 0$ for all n.

$\Sigma = \{$ideals containing no powers of $f\}$.

$(0) \in \Sigma \Rightarrow$ non-empty. Every chain is bounded above, so Σ has a maximal element.

$\mathfrak{N}$ = intersection of all primes

Let P be maximal elmt of Σ.

Claim: P prime.

$xy \in P \Rightarrow x \in P$ or $y \in P$.

OR $x \notin P$ and $y \notin P \Rightarrow xy \notin P$.

$P+(x)$, $P+(y)$ not equal to P

$\Rightarrow f^m \in P+(x)$, $f^n \in P+(y)$. for some m, n.

$\Rightarrow f^{m+n} \in P+(xy) \Rightarrow xy \notin P$. $\Rightarrow \tilde{\mathfrak{N}} = \mathfrak{N}$

Definition: Jacobson Radical $\mathfrak{R}$

Let A be a ring.

$$\mathfrak{R} = \bigcap_{\substack{\mathfrak{m} \subseteq A \\ \text{maximal}}} \mathfrak{m}.$$

intersection of all maximal ideals

Alternative definition

$$x \in \mathfrak{R} \iff 1 - xy \text{ is a unit in } A \text{ for all } y \in A.$$

($\Rightarrow$) Suppose $x \in \mathfrak{R}$, $(1-xy)$ not unit.

$(1-xy) \in \mathfrak{m}$ maximal. $x \in \mathfrak{R} \subseteq \mathfrak{m}$

$\Rightarrow xy \in \mathfrak{m} \Rightarrow (1-xy)+xy = 1 \in \mathfrak{m} \Rightarrow\Leftarrow$.

($\Leftarrow$) Suppose $x \notin \mathfrak{R}$. $\exists\, \mathfrak{m}$ maximal, $x \notin \mathfrak{m}$.

$\mathfrak{m} + (x) = A \Rightarrow 1 = u + xy$, $u \in \mathfrak{m}$.

$1 - xy = u \in \mathfrak{m} \Rightarrow$ not a unit.

$\mathfrak{N} \subseteq \mathfrak{R}$

intersection of all primes

intersection of all maximal ideals.

elements x, $\forall y \in A$ $1-xy$ unit.

x nilpotent $\Rightarrow$ $1-xy$ unit

$$x \quad \frac{1+xy+(xy)^2+\cdots+(xy)^{n-1}}{1-x^ny^n=1.}$$