Commutative Algebra: Artin Rings

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Why the asymmetry?

Given a finite collection of ideals

$\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n \subseteq A$.

- $\mathfrak{a}_k^2 \subset \mathfrak{a}_k$. so too, $\mathfrak{a}_k \supset \mathfrak{a}_k^2 \supset \mathfrak{a}_k^3 \supset \ldots$

- $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_i$ and $\mathfrak{a}_j$.

  $\mathfrak{a}_1 \supset \mathfrak{a}_1\mathfrak{a}_2 \supset \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 \supset \ldots$

Artinian: these chains must be stationary.

Constructions producing inductively larger ideals are not common.

Outline

Every prime is maximal

Let A be an Artin ring.
$\mathfrak{p} \subset A$ prime ideal. Then $\mathfrak{p}$ maximal.

Proof Consider $A/\mathfrak{p}$ integral domain.

Exact sequence of A-mods: $0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$
$\Rightarrow A/\mathfrak{p}$ Artinian.

Take $x \in A/\mathfrak{p}$, $x \neq 0$. $(x) \supset (x^2) \supset (x^3) \supset \cdots$
is stationary. $\exists n$, $(x^n) = (x^{n+1})$. $\Rightarrow x^n = a x^{n+1}$
$\Rightarrow x^n(1 - ax) = 0$. Int domain $\Rightarrow ax = 1$.

Every prime is maximal

$x \neq 0$ unit $\Rightarrow A/\mathfrak{p}$ field

$\Rightarrow \mathfrak{p}$ is a maximal ideal. $\blacksquare$

Nilradical = Jacobson Radical

$$\mathfrak{N} = \bigcap_{\mathfrak{p}\ \text{prime}} \mathfrak{p} = \bigcap_{\mathfrak{p}\ \text{maximal}} \mathfrak{p} \qquad \text{(for Artin Rings)}$$

$$= \mathfrak{R} \quad \text{Jacobson radical.}$$

Finitely many maximals

Let A be an Artin ring. A has finitely many maximal ideals.

Proof $\{m_i\}_{i \in I}$:= set of max'l ideals of A.

Take $\{$ finite intersections $\bigcap_{\substack{j \in J \subset I \\ \text{finite}}} m_j \}$.

By A Artinian, this has a minimal element.

$M = m_1 \cap \dots \cap m_n$. Take m maximal.

Finitely many maximals

$M = \mathfrak{m} \cap M$ by minimality of M.

$\Rightarrow \mathfrak{m} \supset M$. $\Rightarrow \mathfrak{m} \supset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$.

$\Rightarrow \mathfrak{m} \supset \mathfrak{m}_i$ for some i. ($\mathfrak{m}$ prime).

$\Rightarrow \mathfrak{m} = \mathfrak{m}_i$ by maximality of $\mathfrak{m}_i$.

So any maximal ideal of A is in the set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

Nilradical is Nilpotent

Let A be an Artin ring.
$\mathfrak{N} \subset A$ the nilradical. $\exists k$ s.t. $\mathfrak{N}^k = 0$.

<u>Proof</u> Consider $\mathfrak{N} \supset \mathfrak{N}^2 \supset \cdots$ by d.c.c,
this is stationary. $\exists k$ with $\mathfrak{N}^k = \mathfrak{N}^{k+1}$.
Call this $\mathfrak{a}$. WTS: $\mathfrak{a} = 0$.

By way of contradiction: Suppose $\mathfrak{a} \neq 0$.
Then take $\Sigma = \{\mathfrak{b} : \mathfrak{a}\mathfrak{b} \neq 0\}$.
non-empty because $A \in \Sigma$.

Nilradical is Nilpotent

$\Rightarrow \Sigma$ should have a minimal elmt. $\mathfrak{b}$.

$a\mathfrak{b} \neq 0 \Rightarrow \exists x \in \mathfrak{b}$ with $xa \neq 0$.

$\Rightarrow (x)a \neq 0 \Rightarrow (x) \in \Sigma \Rightarrow \mathfrak{b} = (x)$.

Note: $a^2 = \mathfrak{N}^{2k} = \mathfrak{N}^k = a \Rightarrow a^2 = a$.

$xa \neq 0 \Rightarrow xa^2 = xa \neq 0 \Rightarrow (xa)a \neq 0$.

$\Rightarrow xa \in \Sigma$. $xa \subseteq (x) = \mathfrak{b} \Rightarrow xa = (x)$.

$\Rightarrow x = xa,\ a \in a \Rightarrow x = xa^n\ \forall n \geq 1$.

$\Rightarrow x = 0$. $\Rightarrow\!\Leftarrow$.

Def: Krull Dimension

Let $\boxed{\mathfrak{p}_0} \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{\boxed{n}}$ be a chain of prime ideals in A. (finite strictly increasing sequence) The length of the chain is n.

$\dim(A) :=$ Krull dimension of A

supremum of lengths of chains of prime ideals in A.

$A \neq 0$, $\dim(A) \geq 0$. $\dim(A)$ can be ∞.

dim(Artin ring) $= 0$

Every prime is maximal.

$\mathfrak{p}_0 \subset \mathfrak{p}_1 \Rightarrow \mathfrak{p}_0 = \mathfrak{p}_1$ not increasing.

$\mathfrak{p}_0$ is longest possible chain.

$\Rightarrow \dim A = 0$.

Artin ring = dim-0 Noetherian ring

A ring A is Artinian if and only if it is Noetherian of dimension 0.

Proof ($\Rightarrow$) A Artinian $\Rightarrow$ dim 0.

Recall: $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subset \mathfrak{R} = \mathfrak{N} \Rightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_n)^k = 0.$

$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_N = 0.$

each factor is a $(A/\mathfrak{m}_k)$-vector space.

$\Rightarrow$ each factor Artinian $\Leftrightarrow$ Noetherian.

Artin ring = dim-0 Noetherian ring

A Artinian $\Rightarrow$ each factor Artinian $\Rightarrow$ each factor Noetherian $\Rightarrow$ A Noetherian.

($\Leftarrow$) Let A be a Noetherian ring of dim 0.

(0) has finitely many minimal primes.

$\Rightarrow$ finitely many maximal primes.

The intersection of all primes contains $m_1 \cdots m_n \subset \mathfrak{N}$ $\Rightarrow$ $\mathfrak{N}^k = 0$. $\Rightarrow (m_1 \cdots m_n)^k = 0$

Artin ring = dim-0 Noetherian ring

Again, form the composition series:

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_N = 0.$$

Same Reasoning implies A Artinian.

Example: Non-Noetherian dim-0 Local Ring

Take $R = k[x_1, x_2, \dots]$

$S = R/(x_1, x_2^2, x_3^3, \dots)$

This has only one prime: $(x_1, x_2, \dots) =: \mathfrak{m}$

$\mathfrak{p} \subset S$ prime. $x_k^k = 0 \Rightarrow x_k \in \mathfrak{p} \Rightarrow \mathfrak{m} \subset \mathfrak{p}$.

$\mathfrak{m}$ maximal $\Rightarrow \mathfrak{m} = \mathfrak{p}$.

$(x_2) \supset (x_2 x_3) \supset (x_2 x_3 x_4) \supset \dots$ not stationary.

$\mathfrak{m}^n$ in Noetherian Local Ring

Let A be a Noetherian local ring.

Let $\mathfrak{m}$ be its maximal ideal. Then one of the following holds:

1) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ $\forall n$. (A not Artinian)

$\boxed{\text{2) } \mathfrak{m}^n = 0 \text{ for some } n.}$

Proof Suppose $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. $\mathfrak{m} = \mathfrak{R}$.

Nakayama's Lemma: $\mathfrak{m}^{n+1} = \mathfrak{m}(\mathfrak{m}^n) = \mathfrak{m}^n$

$\Rightarrow \mathfrak{m}^n = 0$.

Structure Theorem for Artin Rings

An Artin Ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings.

Artinian ring with unique maximal ideal.

Localize at each of the finitely many maximal ideals, and take product of results:

$$A \simeq \prod_{i=1}^{n} A_{m_i}$$

Proof $\Rightarrow$ Take A Artinian. A has finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$.

Nilradical nilpotent $\Rightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_n)^k = 0$.

$\mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = 0$, $\mathfrak{m}_i + \mathfrak{m}_j = A \quad \forall i \neq j. \Rightarrow \mathfrak{m}_i^k, \mathfrak{m}_j^k$ coprime.

Chinese Remainder Theorem:

$$A \simeq \prod_{i=1}^{n} A/\mathfrak{m}_i^k.$$

Proof $\Leftarrow$ Given $A = \prod_{i=1}^{n} A_i$, each A_i is Artin local ring.

Can we write a different product isomorphic to A?

$A \cong \prod A_i$ given $\mathfrak{a}_i = \ker(\pi_i : A \to A_i)$

$\Rightarrow \mathfrak{a}_i, \mathfrak{a}_j$ coprime $i \neq j$.

Taking $\mathfrak{q}_i \subset A_i$ unique prime. $\mathfrak{p}_i = \pi_i^{-1}(\mathfrak{q}_i)$.

$\mathfrak{a}_i$ is a $\mathfrak{p}_i$-primary ideal. why?.

Proof $\Leftarrow$

$\mathfrak{q}_i^n = 0 \Rightarrow \mathfrak{p}_i^n = \mathfrak{a}_i \Rightarrow \mathfrak{a}_i$ is a power of a maximal ideal $\Rightarrow \mathfrak{a}_i$ is $\mathfrak{p}_i$-primary.

$\bigcap_{i=1}^{n} \mathfrak{a}_i = 0$, $(\mathfrak{a}_i, \mathfrak{a}_j)$ coprime for $i \neq j$.

primary decomposition of zero.

each component isolated.

2nd uniqueness thm for primary decomp: each $\mathfrak{a}_i$ is uniquely determined.

Artin local ring w principal maximal ideal

Let A be local Artin ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$. TFAE:

1. Every ideal in A is principal.
2. The maximal ideal $\mathfrak{m} \subseteq A$ is principal.
3. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$.

1) $\Rightarrow$ 2) $\Rightarrow$ 3) $m = (x)$ m/m^2 killed by m.

$\Rightarrow$ 1-dim vectorspace over A/m.

3) $\Rightarrow$ 1) i) $\dim_k(m/m^2) = 0$. $\Rightarrow$ $m = m^2$.

$\Rightarrow m(m) = m \Rightarrow m = 0$. ($m = \mathfrak{R}$, Nakayama)

ii) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1. \Rightarrow \mathfrak{m} = (x).$

Take $\mathfrak{a} \subset A.$ $\mathfrak{a} \neq 0$, $\mathfrak{a} \neq A$

$\Rightarrow \exists r$ s.t. $\mathfrak{a} \subset \mathfrak{m}^r$, $\mathfrak{a} \not\subset \mathfrak{m}^{r+1}.$

$\exists y \in \mathfrak{a}$ s.t. $y = a x^r$, but $y \notin (x^{r+1}).$

$\Rightarrow a \notin (x) \Rightarrow a$ unit. $\Rightarrow x^r = a^{-1} y \in \mathfrak{a}.$

$\Rightarrow \mathfrak{a} \supseteq (x^r) = \mathfrak{m}^r \Rightarrow \mathfrak{a} = \mathfrak{m}^r = (x^r).$

Examples

- $\mathbb{Z}/(p^n)$. $\quad p\mathbb{Z}/(p^n)$ maximal ideal.

  Every ideal in here is principal,

  $$\dim_{\mathbb{Z}/p\mathbb{Z}}\left(p\mathbb{Z}/p^n \big/ p^2\mathbb{Z}/p^n\right) = \dim(\mathbb{Z}/p\mathbb{Z}) = 1.$$

- $k[x^2, x^3]/(x^4)$.

  (x^2, x^3) is the unique maximal ideal, not principal.

$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}$ has dim 2 over k.