Commutative Algebra: Graded Rings & Modules

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Outline

Graded Rings & Modules

Poincaré Series

Hilbert Function

Def: Graded Ring

A *graded ring* is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A with

$$A = \bigoplus_{n=0}^{\infty} A_n, \qquad A_m A_n \subseteq A_{m+n}.$$

Note: A_0 is a subring of A and each A_n is an A_0-module.

$A_0 A_0 \subseteq A_{0+0} = A_0.$

$A_0 A_n \subseteq A_{n+0} = A_n.$

Examples: Graded Rings

Trivial. $A = A_0$.

$n > 0$, $A_n = 0$.

$\bigoplus A_n = A_0 = A$ ✓ $\quad A_m A_n \doteq \underset{\displaystyle A_{m+n}}{\overset{\shortparallel}{0}}$ if $m > 0$ or $n > 0$.

$A = k[x_1, \ldots, x_n]$.

A_n = degree n homogeneous polynomials in A.

homog: $x_1^2 x_2 + x_3^3 - x_1 x_2 x_3$.

non-homog: $x_1^3 + x_1 x_2$.

$A = \bigoplus_{n=0}^{\infty} A_n \rightsquigarrow f \to f_n + f_{n-1} + \cdots + f_0$. $\quad A_m A_n \subseteq A_{m+n}$.

Def: Graded Modules

Let A be a graded ring. A *graded* A-**module** is an A-**module** M together with a family $(M_n)_{n\geq 0}$ of subgroups of M with

m-graded part of A

$$M = \bigoplus_{n=0}^{\infty} M_n, \qquad A_m M_n \subseteq M_{m+n}.$$

Note: M_n is an A_0-module for all n.

$$A_0 M_n \subseteq M_{n+0} = M_n.$$

Def: Homogeneous Element

A homogeneous element in a graded module M is an elmt of M_n for some n.

Given $x \in M$, we can write

$$x = \sum_{i \in I} x_i \quad , \quad I \subseteq \mathbb{N} \text{ finite index set}$$

with $x_i \in M_i$ homogeneous.

Thm: Noetherian graded rings

Let A be a graded ring. The following are equivalent:

1. A is a Noetherian ring.
2. A_0 Noetherian, A finitely-generated as an A_0-algebra.

2) $\Rightarrow$ 1) Hilbert Basis Thm.

1) $\Rightarrow$ 2) A Noetherian. Let $A_+ = \bigoplus_{n>0} A_n$. A_+ is an ideal.

$A_0 = A/A_+ \leftarrow$ Noetherian.

Proof

A fin. gen. as an A_0-algebra.

A_+ is a finitely-generated ideal.

$\Rightarrow A_+ = \langle x_1, \ldots, x_r \rangle$.

Let $A' = A_0[x_1, \ldots, x_r] \subseteq A$.

Claim: $A_n \subseteq A'$ for all n. Argue by induction.

Base step: $A_0 \subseteq A'$ ✓

Proof

Assume $A_k \subseteq A'$ for $k < n$.

Consider $x \in A_n \subseteq A_+ \Rightarrow x \in A_+$

$\Rightarrow x = \sum_{i=1}^{r} a_i x_i$, $a_i \in A$.

$x_i \in A_+$ for all $i \Rightarrow a_i \in A_{n-k_i}$

where k_i is the grading of x_i.

$\Rightarrow a_i \in A'$. $\Rightarrow x \in A'$. $\boxtimes$

Henri Poincaré

Def: Poincaré Series

- A = Noetherian graded ring generated over A_0 by elements $x_1, \ldots, x_s$ in degrees $k_1, \ldots, k_s$,
- M a finitely-generated graded A-module.
- λ an additive function on f.g. A_0-modules.

Then, $P(M, t)$ is defined as $\sum_{n \geq 0} \lambda(M_n) t^n$.

(Hilbert-) "Poincaré series".

$\hookrightarrow A = k$-alg.

$\lambda(M_n) = \dim_k M_n$.

Examples

Trivial Graded Ring. $A_0 = A, \; A_n = 0 \quad n > 0.$

$$\lambda(A)t^0 + 0t^1 + \dots = \lambda(A) \quad (\text{e.g. } \underset{\sim}{1})$$

$k[x]$ $\quad \lambda(M_n) = \dim_k M_n$

$$k[x] = k \oplus kx \oplus kx^2 \oplus \cdots$$

$$\Rightarrow P(k[x], t) = 1 + t + t^2 + \dots = \frac{1}{1-t}.$$

Example

A

$\shortparallel$

$k[x_1, \ldots, x_n]$ For degree s, the monomials

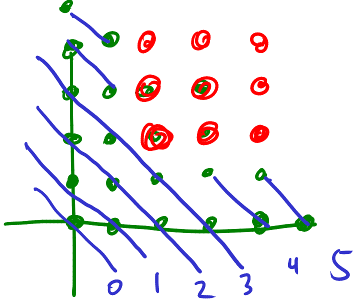
$x_1^{a_1} \cdots x_n^{a_n}$ where $a_1 + \cdots + a_n = s$ span

A_s. $|\ast|\ \ast||\ast \quad \ast$

$$\#\text{monomials} = \binom{n+s-1}{n-1}.$$

$$P(A,t) = \sum_{s=0}^{\infty} \binom{n+s-1}{n-1} t^s = \frac{1}{(1-t)^s}.$$

Example



$M = k[x,y]/(x^2y^2)$

$$= \sum_{n=0}^{\infty} (\dim_k M_n)\, t^n = 1 + 2t + 3t^2 + 4t^3 + 4t^4 + 4t^5 + 4t^6 + \cdots$$

$$P(k[x,y],t) = \frac{1}{(1-t)^2}.$$

$$P(x^2y^2k[x,y],t) = \frac{t^4}{(1-t)^2} \Rightarrow P(M,t) = \frac{1-t^4}{(1-t)^2}.$$

Thm: Poincaré Series is Rational

- $A =$ Noetherian graded ring generated over A_0 by elements $x_1, \ldots, x_s$ in degrees $k_1, \ldots, k_s$,
- M a finitely-generated graded A-module.
- λ an additive function on f.g. A_0-modules.

Then, $P(M,t) = \sum_{n \geq 0} \lambda(M_n) t^n = \dfrac{f(t)}{\prod_{i=1}^{s}(1 - t^{k_i})}$, where $f(t) \in \mathbb{Z}[t]$.

Proof

Induction # generators s of A over A_0.

Base case: $s=0$. $A=A_0$.

M f.g. A-module $\Rightarrow$ M f.g. A_0-mod.

$\Rightarrow M = \bigoplus_{i=1}^{n} A_0 x_i \Rightarrow P(M,t)$ is polynomial.

Induction Step: Suppose true for $r<s$. WTS: true for s generators.

Proof

Consider exact sequence:

$$0 \to K_n \to M_n \xrightarrow{\cdot x_s} M_{n+k_s} \longrightarrow L_{n+k_s} \to 0$$

λ additive function:

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0.$$

$$P(K,t) - P(M,t) + t^{k_s} \underset{\sim}{P(M,t)} - t^{k_s} \underset{\sim}{P(L,t)}$$

$$+ g(t) = 0.$$

Proof

$$(1-t^{k_s})P(M,t) = P(K,t) - t^{k_s}P(L,t) + g(t).$$

K, L are both annihilated by x_s

so they are modules over $A_0[x_1, \ldots, x_{s-1}]$

$\Rightarrow$ $P(K,t)$ and $P(L,t)$ can be written

as $\dfrac{h(t)}{\prod_{i=1}^{s-1}(1-t^{k_i})}$. $\Rightarrow$ $P(M,t)$ has the

desired form.

Def: Hilbert Function

Let A be a Noeth. graded ring,
M fin-gen graded A-module,
λ additive function on the class
of f.g. A_0-modules.

$\lambda(M_n)$ as a function of n
is called the "Hilbert function" of M.

Examples

Trivial Graded Ring. $A = k$

$\lambda(M_n) = \dim_k M_n$. $f(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{else}. \end{cases}$

$k[x]$

$\lambda(M_n) = \dim_k M_n$. $f(n) = 1$.

Example

$k[x_1, \ldots, x_n]$

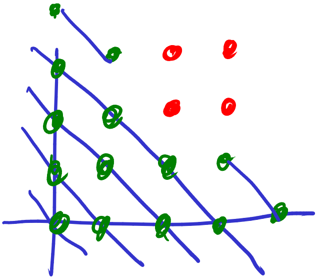
s

$\lambda(M_n) = \dim_k M_n$ = # monomials of degree n in s variables.

$$= \binom{s+n-1}{s-1}$$

← polynomial function in n.

Example



$k[x, y]/(x^2y^2)$

$\lambda(M_n) = \dim_k(M_n)$

$$f(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 3 & n = 2 \\ 4 & n \geq 3 \end{cases}$$

Thm: Hilbert function is Polynomial

eventually

- A Noetherian graded ring generated over A_0 by d elements of degree 1,
- M a finitely-generated graded A-module.
- λ an additive function on f.g. A_0-modules.

Then, for $n \gg 0$, $\lambda(M_n)$ is a polynomial in n of degree $d - 1$.

Proof

$$P(m,t) = \frac{f(t)}{(1-t)^s}.$$

$$f(t) = \sum_{k=0}^{m} a_k t^k \qquad \frac{1}{(1-t)^s} = \sum_{d=0}^{\infty} \binom{d+s-1}{s-1} t^d.$$

$$\lambda(M_n) = \text{Coeff of } t^n$$

$$= \sum_{k+d=n} a_k \binom{d+s-1}{s-1} = \sum_{k=0 \quad m}^{n} a_k \binom{n-k+s-1}{s-1}.$$

If $n<m$, the function does not include all a_k.

Proof

Once, we have $n \geqslant m$, all a_x's are included and the Hilbert function is fixed as a polynomial.