Commutative Algebra: Two Ideal Theorems

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The Two Theorems on Ideals

Chinese Remainder Theorem

Prime Avoidance Lemma

Defns: Direct Product, Coprime Ideals

Let $A_1, \dots, A_n$ be rings.

$A_1 \times \dots \times A_n = \{(x_1, \dots, x_n) \mid x_i \in A_i\}$

with componentwise multiplication & addition. Called "direct product".

Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ ideals.

$\mathfrak{a}, \mathfrak{b}$ are coprime if $\mathfrak{a} + \mathfrak{b} = (1)$.

Chinese Remainder Theorem

Proposition 1.10 (Atiyah-MacDonald)

Let A be a ring with $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subseteq A$ ideals. Define a homomorphism $\phi : A \to \prod_{i=1}^{n}(A/\mathfrak{a}_i)$ by the rule $\phi(x) = (x + \mathfrak{a}_1, \ldots, x + \mathfrak{a}_n)$.

1. If $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$.
2. ϕ is surjective $\iff$ $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$.
3. ϕ is injective $\iff$ $\bigcap \mathfrak{a}_i = (0)$.

Proof: $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$

Consider $n=2$. $\mathfrak{a}_1, \mathfrak{a}_2$ coprime.

WTS: $\mathfrak{a}_1 \mathfrak{a}_2 = \mathfrak{a}_1 \cap \mathfrak{a}_2$.

($\subseteq$) $\mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_1$ by closure under A-multiplication.

similarly, $\mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_2$

$\Rightarrow \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_1 \cap \mathfrak{a}_2$.

($\supseteq$) $\mathfrak{a}_1 + \mathfrak{a}_2 = (1)$.

$(\mathfrak{a}_1 \cap \mathfrak{a}_2)(\mathfrak{a}_1 + \mathfrak{a}_2) = (\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{a}_1 + (\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{a}_2$

Proof: $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$

$$(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{a}_1 + (\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{a}_2 \subseteq \mathfrak{a}_1 \mathfrak{a}_2.$$

$$\Rightarrow (\mathfrak{a}_1 \cap \mathfrak{a}_2) \subseteq \mathfrak{a}_1 \mathfrak{a}_2$$

Let $n > 2$.

$$\prod^{n} \mathfrak{a}_i = \left(\prod^{n-1} \mathfrak{a}_i\right)\mathfrak{a}_n, \quad \bigcap^{n} \mathfrak{a}_i = \bigcap^{n-1} \mathfrak{a}_i \cap \mathfrak{a}_n$$

Apply induction hypothesis

$$\mathfrak{b} = \prod^{n-1} \mathfrak{a}_i = \bigcap^{n-1} \mathfrak{a}_i.$$

WTS: $\mathfrak{a}_n \mathfrak{b} = \mathfrak{a}_n \cap \mathfrak{b}$, sufficient: $\mathfrak{a}_n, \mathfrak{b}$ coprime.

Proof: $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$

$\mathfrak{b} = \prod^{n-1} \mathfrak{a}_i$ For each i, $\exists x_i \in \mathfrak{a}_i, y_i \in \mathfrak{a}_n$

such that $x_i + y_i = 1$. $y_i \equiv 1 - x_i \ (\mathfrak{a}_i)$

$\Rightarrow \prod y_i \in \mathfrak{a}_n \equiv 1 \bmod \mathfrak{b}$.

$\Rightarrow \mathfrak{a}_n + \mathfrak{b} = (1) \Rightarrow$ coprime.

Proof: ϕ is surjective $\iff$ $\mathfrak{a}_i, \mathfrak{a}_j$ coprime

$(\Rightarrow)$ $(1, 0, \ldots, 0) \in \mathrm{Im}(\phi)$

$\Rightarrow \exists x \quad x \equiv 1 \mod a_1, \; x \equiv 0 \mod a_2.$

$(1-x) \in a_1 \qquad x \in a_2$

$\Rightarrow (1-x) + x = 1 \in a_1 + a_2 \Rightarrow$ coprime

$\Rightarrow a_i, a_j$ coprime for $i \neq j$.

$(\Leftarrow)$ Enough to find some x s.t.

$\phi(x) = (1, 0, \ldots, 0)$

$\exists u_i \in a_1, \; v_i \in a_i \quad 2 \leq i \leq n$

Proof: ϕ is surjective $\iff$ $\mathfrak{a}_i, \mathfrak{a}_j$ coprime

such that $u_i + v_i = 1$.

$v_i = 1 - u_i \Rightarrow v_i \equiv 1 \ (\mathfrak{a}_1)$.

$\prod_{i=2} v_i \equiv 1 \ (\mathfrak{a}_1)$ and $\equiv 0 \ (\mathfrak{a}_i)$ $i \geq 2$.

$\Rightarrow \phi(\prod_{i=2}^{n} v_i) = (1, 0, \ldots, 0)$.

Proof: ϕ is injective $\iff \bigcap \mathfrak{a}_i = 0$

ϕ injective $\iff \ker(\phi) = 0.$

$\ker(\phi) = \{x \mid x \in a_i, 1 \le i \le n\}$

$= \bigcap a_i = 0.$

Application: $\mathbb{Z}$

$$27720 = 5 \times 7 \times 8 \times 9 \times 11$$
$$= 2^3 \times 3^2 \times 5 \times 7 \times 11$$

$$\mathbb{Z}/27720\mathbb{Z} \simeq \mathbb{Z}/8 \times \mathbb{Z}/9 \times \mathbb{Z}/5 \times \mathbb{Z}/7 \times \mathbb{Z}/11.$$

$$N \leftarrow (n_1, n_2, n_3, n_4, n_5)$$

CS Applications.

Application: $k[x]$, field k.

$$k[x] / \prod_{i=1}^{n} (x - a_i) \simeq$$

$$k[x]/(x-a_1) \times \cdots \times k[x]/(x-a_n)$$

$$p(x) \longleftarrow (b_1, \ldots, b_n) \in k^n$$

Lagrange interpolation.

Prime Avoidance Lemma

Proposition 1.11(i) (Atiyah-MacDonald)

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals and let $\mathfrak{a}$ be an ideal contained in $\bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

A Sharper Version

Lemma 3.3 (Eisenbud)

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subseteq A$ be ~~prime~~ ideals and let $\mathfrak{a}$ be an ideal contained in $\bigcup_{i=1}^{n} \mathfrak{p}_i$. Suppose either that:

1. A contains an infinite field, OR
2. all but two of the $\mathfrak{p}_i$ are prime.

Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

Proof: Infinite field

<u>Lemma</u> A vector space over an inf. field cannot be a finite union of proper subspaces.

Given $k \subseteq A$, every ideal is a k-vector space, so $\mathfrak{a}$ must be inside one of the $\mathfrak{p}_i$.

Proof: Infinite field

Proof of Lemma: Suppose $V = \bigcup_{i=1}^{n} W_i$

s.t. no W_i can be omitted.

$\exists x \in W_1$, s.t. $x \notin W_i$ $i \neq 1$.

$\exists y \in V \setminus W_1$. Consider $\{x + \alpha y : \alpha \in k\}$.

This is contained in $V \Rightarrow$ each is in

some $W_i \Rightarrow x + \alpha_1 y,\ x + \alpha_2 y \in W_j$ $\overset{j \neq 1}{\alpha_1 \neq \alpha_2}$

$\alpha_2(x+\alpha_1 y) - \alpha_1(x + \alpha_2 y) = (\alpha_2 - \alpha_1)x \in W_j \Rightarrow \Leftarrow$.

Proof: all but two $\mathfrak{p}_i$ are prime

$n=1$: $\mathfrak{a} \subseteq \bigcup_{i=1} \mathfrak{p}_i \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}_1$.

Induction: assume all $\mathfrak{p}_i$'s are necessary so no proper subunion contains $\mathfrak{a}$.

$n=2$: $\mathfrak{a} \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$.

$x_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$

$x_2 \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$

$\Rightarrow x_1 + x_2 \notin \mathfrak{p}_1, \mathfrak{p}_2 \Rightarrow x_1 + x_2 \notin \mathfrak{a}$

$\Rightarrow\Leftarrow$.

Proof: all but two $\mathfrak{p}_i$ are prime

$n > 2$: Suppose $\mathfrak{p}_1$ is prime.

As before, there is $x_i \in \mathfrak{p}_i$ s.t.

$x_i \notin \mathfrak{p}_j$ for all $j \neq i$.

$$y = \underset{\mathfrak{p}_1}{\underset{\uparrow}{x_1}} + \underbrace{x_2 x_3 \cdots x_n}$$

not in $\mathfrak{p}_1$, because $\mathfrak{p}_1$ is prime,

So, $y \notin \bigcup \mathfrak{p}_i \Rightarrow \mathfrak{a}$ not ideal $\Rightarrow\Leftarrow$.

Application

Let R be a Noetherian ring, M a finitely generated nonzero R-module. Every ideal consisting entirely of zerodivisors on a module M annihilates some element of M.

Idea of Proof

The zerodivisors are contained in the union of "associated primes" of a module. So this ideal must be contained in one such associated prime.