Commutative Algebra: Spec

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Spec: From Rings to Top Spaces

A Topology on the Primes

Examples of Spectra

Functoriality

Much of this material developed from Exercises 15-26 in Chapter 1 of Atiyah-MacDonald.

Definition: Topological Space

(X, τ): X is a set.
τ is a collection of subsets of X satisfying:

1) $\phi, X \in \tau$.
2) Arbitrary intersections of elmts of τ are in τ.
3) Finite unions of elmts of τ are in τ.

$\tau :=$ "closed sets of the topology".

Examples of Topological Spaces

For any set $X \neq \emptyset$,

1) Trivial topology: $\tau = \{\emptyset, X\}$.

2) Discrete topology: $\tau = \mathcal{P}(X)$.

For $X = \mathbb{R}^n$,

3) Euclidean topology:
closed balls around points are in τ

Closed Sets: $V(E)$

Let R be a ring.

$E \subseteq R$ subset.

$V(E)$ = set of prime ideals containing E.

$$\begin{cases} \operatorname{Spec} R = \{\text{prime ideals of } R\} \\ \tau = \{V(E) : E \subseteq R\}. \end{cases}$$

defines a topological space.

Satisfying Axioms

$$\{V(E) : E \subseteq R\} =: \tau$$

1) $\phi, X \in \tau$. $\phi = V(1)$.

$X = V(0)$.

2) Arbitrary intersections?

$$\bigcap_{i \in I} V(E_i) = V\left(\sum_{i \in I}(E_i)\right).$$

3) Finite unions? $\bigcup_{i=1}^{n} V(E_i)$

$$= V\left(\prod_{i=1}^{n}(E_i)\right).$$

Basic Open Sets

* Open sets are complements of closed sets.
* Basis for the open sets: $\mathcal{B}$, such that any open set can be written as a union of elements of $\mathcal{B}$.

$$D(f) = \{ \mathfrak{p} : f \notin \mathfrak{p} \} \text{ for } f \in R.$$

$$= \operatorname{Spec} R \setminus V(f)$$

"distinguished open of f"

Basic Open Sets

Take $U \subset \operatorname{Spec} R$ open.

$\Rightarrow U = \operatorname{Spec} R \setminus V(I)$, $I \subseteq R$.

For any $x \in U$, $I \not\subseteq x$. $\Rightarrow \exists f_x$ s.t.

$f_x \in I$ and $f_x \notin x$. $\Rightarrow x \in D(f_x)$.

$\hookrightarrow V(f_x) \supseteq V(I) \Rightarrow D(f_x) \subseteq U$.

$$\Rightarrow U = \bigcup_{x \in U} D(f_x).$$

Spec(k), k field

Only ideals of a field are (0), k.

Only (0) is prime.

(0) $\bullet$ Spec k.

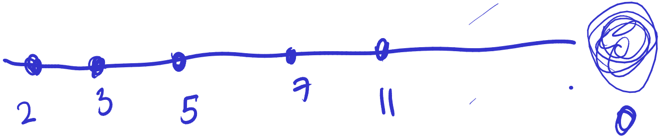
Spec($\mathbb{Z}$)

prime ideals: $\{(p) : p \text{ prime}\} \cup \{0\}$.

$V(p) = \{(p)\} \leftarrow$ closed point.

$V(0) = \operatorname{Spec} \mathbb{Z}$. $\quad 0 \leftarrow$ not closed.

$V(n) = \{(p) : p \mid n\} \longleftarrow$ finite collection of primes.



Spec($\mathbb{C}[x]$)

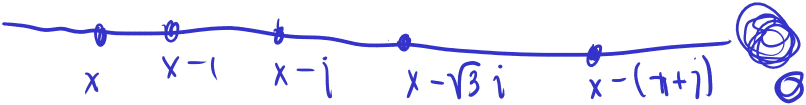
F.T. Alg: every polynomial over $\mathbb{C}$ factors into linear polynomials.

$\{(x-a) : a \in \mathbb{C}\} \cup \{0\}$.

$V(x-a) = \{(x-a)\}$ ← closed point.

$V(0) = \operatorname{Spec} \mathbb{C}[x]$ ← not closed.

$V(p(x)) = \{(x-a) : a = \text{root of } p(x)$, equiv, $p(a) = 0\}$



Spec($\mathbb{C}[x, y]$)

Prime ideals?

1) 0.

2) $\{(x-a, y-b) : (a,b) \in \mathbb{C}^2\}$ ← maximal

3) $\{$(irreducible polynomials over $\mathbb{C}$ in x and y)$\}$

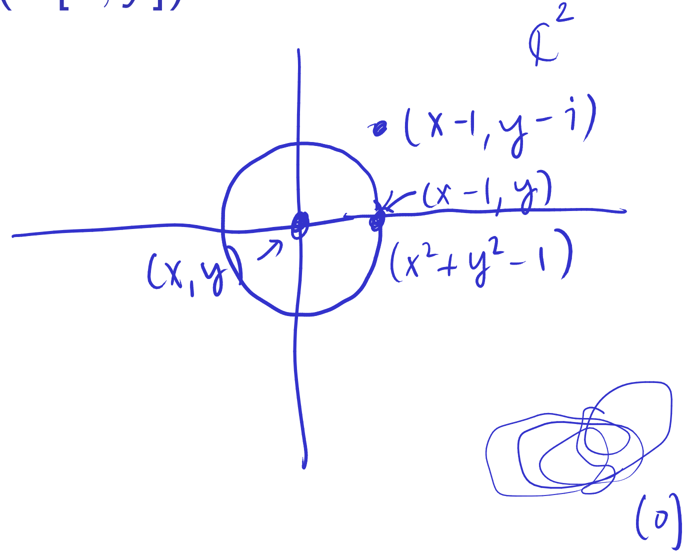
x^2+y^2-1, y^2-x^3-1

maximal = closed points.

zero ⇝ generic point for all of $\mathbb{C}^2$.

$(f : f$ irred$)$ ← closure will include all ideals $(x-a, y-b)$ s.t. $f(a,b)=0$.

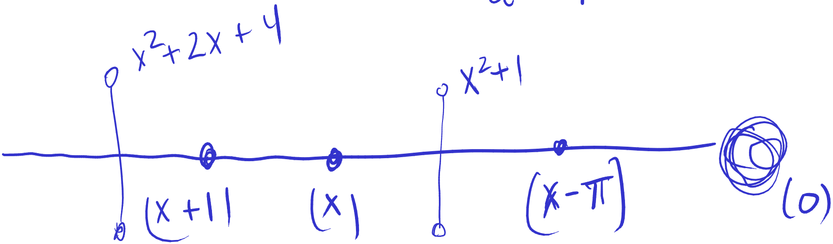
Spec($\mathbb{C}[x, y]$)



Spec($\mathbb{R}[x]$)

Prime ideals: $\{(0)\} \cup \{(x-a) : a \in \mathbb{R}\}$

$\cup \{(x^2+ax+b : \text{irreducible } a^2-4b<0)\}$



(0) is not closed, but the others are.

Definition: Category

A category $\mathscr{C} = (\mathrm{Obj}(\mathscr{C}), \mathrm{Mor}(\mathscr{C}))$

"objects" "morphisms"

satisfying: 1) $f \in \mathrm{Mor}(\mathscr{C})$ has a domain and a codomain in $\mathrm{Obj}(\mathscr{C})$ (i.e. $f : A \to B$).

2) $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$ then there exists composition $g \circ f \in \mathrm{Mor}(A, C)$.

Definition: Category

3) associativity of composition:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

4) $\exists X \in \mathrm{Obj}(\mathcal{C})$ there is a map in $\mathrm{Mor}(X, X)$ called id_X.

$$f \circ id_X = f, \quad id_X \circ g = g.$$

~~Definition:~~ Category Examples

1) Set : obj = sets, mor = set maps.

2) Group: obj = groups, mor = gp hom's.

3) Ring. obj = rings, mor = ring hom's.

4) Top. obj = topological spaces,
mor = continuous maps
(pre-image of open is open)

Definition: Functor

A map $\mathcal{F}$ from category $\mathcal{C}$ to category $\mathcal{D}$ is a functor if

1) $\mathcal{F}: \mathrm{Obj}(\mathcal{C}) \to \mathrm{obj}(\mathcal{D})$.

2) $\forall X, Y \in \mathcal{C}$, $\mathcal{F}: \mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathcal{F}(X), \mathcal{F}(Y))$

   i) $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$

   ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$. "covariant"

Definition: Functor

A contravariant functor sends

$$\mathcal{F} : \mathrm{mor}(X, Y) \to \mathrm{mor}(\mathcal{F}(Y), \mathcal{F}(X)).$$

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Examples of Functors

1) Forgetful functor:

Group $\to$ Set (treat each group as a set)

Ring $\to$ Group (just deal with addition)

2) Fundamental group:

Pointed top sp $\longrightarrow$ Group

continuous map $\rightsquigarrow$ group homomorphism.

Spec is a functor $\mathrm{Ring} \to \mathrm{Top}$

$f : R \longrightarrow S$ ring homomorphism

$\varphi_f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ contravariant

$\mathfrak{p} \longmapsto f^{-1}(\mathfrak{p})$

continuous?

$D(x) = \{ \mathfrak{p} \in \operatorname{Spec} R : x \notin \mathfrak{p} \}$

$\varphi_f^{-1}(D(x)) = D(f(x)) = \{ \mathfrak{q} \in \operatorname{Spec} S : f(x) \notin \mathfrak{q} \}.$