Commutative Algebra: Intro to Modules

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Intro to Modules

First Definitions

Homomorphisms & Quotients

Operations on Submodules / Modules

Definition: Module

Let A be a ring. Let M be an abelian group. Let $\varphi : A \times M \to M$ be a map satisfying:

1) $a \cdot (m_1 +_M m_2) = a \cdot m_1 + a \cdot m_2$ $\quad \forall a \in A$
   $\forall m_1, m_2 \in M$.
2) $(a_1 a_2) \cdot m = a_1 \cdot (a_2 \cdot m)$ $\quad \forall a_1, a_2 \in A,\ m \in M$.
3) $(a_1 +_A a_2) \cdot m = a_1 \cdot m + a_2 \cdot m$ $\quad$ "
4) $1_A \cdot x = x$ $\quad \forall x \in M$.

Definition: Module

Let A be a ring.

Let M be an abelian group.

Define $\varphi : A \longrightarrow \mathrm{End}(M, M)$

ring homomorphism, then uniquely determines multiplication.

$$\underset{A}{a} \cdot \underset{M}{x} = \underset{\mathrm{End}(M,M)}{\varphi(a)}(x) = \underset{M}{y}$$

$1_A \cdot x = x$ necessary?

$$(a_1 a_2)x = a_1(a_2 x)$$

$$(1_A \cdot 1_A)x = 1_A(1_A x)$$

$$\shortparallel$$

$$1_A x = 1_A(1_A x)$$

<u>Ex</u> module k^2, k field.

$$1_A \cdot (x_1, x_2) = (x_1, 0)$$

Therefore, the last axiom is necessary!

Examples: Ideals, Abelian Groups, Vector Spaces

1) $\mathfrak{a} \subset A$ ideal. use std multiplication.

Since $\mathfrak{a}$ is closed under A-mult, this is well defined.

2) G abelian gp. G is a $\mathbb{Z}$-module

$$ng = \underbrace{g + \cdots + g}_{n \text{ times}} \qquad -ng = \underbrace{(-g) + \cdots + (-g)}_{n \text{ times}}.$$

3) k field. V k-vector space.

$\Rightarrow$ V is a k-module.

Example: $k[x]$-modules / k-vector spaces with linear map

Let V be a $k[x]$-module. $a_i \in k$.

$$p(x)\cdot v = (a_n x^n + \cdots + a_1 x + a_0)\cdot v \in V$$

$$= a_n x^n \cdot v + \cdots + a_1 x \cdot v + a_0 \cdot v.$$

$$= a_n x(\cdots(x\cdot v)) + \cdots + a_1 x\cdot v + a_0\cdot v.$$

$x \cdot v$ defines a linear map $V \to V$.

$\Rightarrow p(x)$ also a linear map.

Example: $k[G]$-modules / k-representations of G

Recall: $k[G]$ not commutative for G non-abelian.

$(a_1 g_1 + \cdots + a_n g_n) \cdot x$ $\quad x \in M, \quad a_i \in k$

$a_1(g_1 \cdot x) + \cdots + a_n(g_n \cdot x)$

$\varphi : k[G] \longrightarrow \mathrm{End}(M, M) \leftarrow \mathrm{Aut}(M)?$

group representation.

g_i's have inverses.

Definition: Submodule

Let A be a ring, M A-module.

$N \subseteq M$ is a submodule if

N is a subgroup and closed under

multiplication by A.

Defn: Module Homomorphism

Let M_1, M_2 be A-modules.

Then, $\varphi : M_1 \to M_2$ is a module homomorphism if

1) $\varphi(x+y) = \varphi(x) + \varphi(y)$.

2) $\varphi(ax) = a\varphi(x)$.

Defn: Kernel, Image

Let $\varphi : M \to N$ be an A-module homomorphism. Then,

- $\ker(\varphi) = \{x \in M : \varphi(x) = 0\}$ is a submodule of M.

  Let $x, y \in \ker(\varphi) \Rightarrow \varphi(x+y) = \varphi(x) + \varphi(y) = 0$.

  $a \in A$, $x \in \ker(\varphi) \Rightarrow \varphi(ax) = a\varphi(x) = 0$.

- $\mathrm{Im}(\varphi) = \{y \in N : \exists x, \text{ s.t. } \varphi(x) = y\}$. also a submodule.

  $x, y \in \mathrm{Im}(\varphi) \Rightarrow \exists v, w$ s.t. $\varphi(v) = x$, $\varphi(w) = y$

  $\varphi(v+w) = x + y$.

  $a \in A$, $x \in \mathrm{Im}(\varphi) \Rightarrow \exists v$ s.t. $\varphi(v) = x \Rightarrow \varphi(av) = ax$.

Definition: Module Quotient

- If $N \subseteq M$ is an A-submodule, we can define M/N as the quotient group, which is well-defined as an A-module.

$$x + N = x' + N \iff x = x' + n, \quad n \in N$$

$$ax = ax' + an \implies ax \equiv ax' \mod N.$$

Defn: Cokernel

Let $\varphi : M \to N$ A-modules.

Define cokernel:

$$\mathrm{Coker}(\varphi) = N/\mathrm{im}(\varphi).$$

Isomorphism Theorems of Groups apply to Modules:

$$M/\ker(\varphi) \cong \mathrm{im}(\varphi).$$

Example: $2\mathbb{Z} \to \mathbb{Z}/32\mathbb{Z}$, $2 \mapsto \bar{4}$

homomorphism of $\mathbb{Z}$-modules.

$\ker(\varphi) = 16\mathbb{Z}$.

$\mathrm{im}(\varphi) = 4\mathbb{Z}/32\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z}$.

$$\mathrm{Coker}(\varphi) = (\mathbb{Z}/32\mathbb{Z})/(4\mathbb{Z}/32\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}.$$

$$M/\ker(\varphi) = 2\mathbb{Z}/16\mathbb{Z} \cong 4\mathbb{Z}/32\mathbb{Z} = \mathrm{Im}(\varphi)$$

Sum of Submodules & Finitely-Generated Modules

- Let $M_1, \dots, M_n \subset N$ be submodules of an A-module.

  Define $\sum_i M_i = \{ \sum_{i=1}^n x_i : x_i \in M_i \}$

- $M = Ax = \{ rx : r \in A \}$ is called a cyclic module.
- If M is a sum of cyclic modules, it is called "finitely-generated".

Intersection, Product (by Ideal)

Let $M_1, \dots, M_n \subseteq N$ submodules.

$\bigcap_{i=1}^{n} M_i =$ set-theoretic intersection, which is a submodule.

Product: Let $\mathfrak{a} \subseteq A$ be an ideal, and M an A-module.

$N = \mathfrak{a}M$ is an A-module.

Defn: Colon Ideal

Recall that $\mathfrak{a}, \mathfrak{b} \subseteq A$ ideals

$\Rightarrow \mathfrak{a} : \mathfrak{b} = \{ x \in A : x\mathfrak{b} \subseteq \mathfrak{a} \}$.

For $M \subseteq N$ submodule, A-module.

Define $M : N = \{ x \in A : xN \subseteq M \}$.

Defn: Annihilator, Faithful Module

If $M = 0$, then

$0 : N$ is denoted $\operatorname{ann}(N)$, the annihilator of N.

<u>Ex</u> $N = \mathbb{Z}/32\mathbb{Z}$ as $\mathbb{Z}$-module.

$\Rightarrow \operatorname{ann}(N) = 32\mathbb{Z}$.

If M has $\operatorname{ann}(M) = 0$ then M is called a faithful module.

Defn: Direct Sum & Direct Product

Let $\{M_i\}_{i\in I}$ be A-modules, then

$$\bigoplus_{i\in I} M_i = \{\sum_{i\in I} x_i : x_i \in M_i, \text{ all but finitely many } x_i = 0\}$$

$$\prod_{i\in I} M_i = \{(x_i)_{i\in I} : x_i \in M_i\}$$

Equal when I finite index set.

$I = \mathbb{Z}$: cardinality of $\bigoplus_{i\in I} M_i$ is countable, card. of $\prod_{i\in I} M_i$ uncountable.