Commutative Algebra: Exact Sequences & Hom

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Outline

Definition: Exact Sequence

Let $\{M_i\}$, $i = 1, \ldots, n$ be a sequence of A-modules, with A-module homomorphism $f_i : M_i \to M_{i+1}$.

Then,

$$\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$$

is an exact sequence if for all i

$\mathrm{Ker}(f_{i+1}) = \mathrm{Im}(f_i)$.

Short Exact Sequences

$$0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} M_3 \xrightarrow{f_4} 0$$

- f_2 injective.
- f_3 surjective.
- $\mathrm{Coker}(f_2) \simeq M_2/\ker(f_3)$.

Long → Short Exact Sequences

$$0 \to M_1 \to M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} M_5 \to 0$$

$$0 \to M_1 \to M_2 \to \mathrm{Im}(f_2) \to 0$$

$$0 \to \mathrm{Im}(f_2) \to M_3 \xrightarrow{f_3} \mathrm{Im}(f_3) \to 0$$

$$0 \to \mathrm{Im}(f_3) \to M_4 \to M_5 \to 0$$

1 long exact $\rightsquigarrow$ 3 short exact.

Proposition 2.10 (Atiyah-MacDonald)

Let the following diagram commute, with rows exact:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0
\end{tikzcd}$$

Then there is an exact sequence:

$$0 \to \operatorname{Ker} f' \xrightarrow{\bar{u}} \operatorname{Ker} f \xrightarrow{\bar{v}} \operatorname{Ker} f'' \xrightarrow{\delta}$$
$$\to \operatorname{Coker} f' \xrightarrow{\bar{u}'} \operatorname{Coker} f \xrightarrow{\bar{v}'} \operatorname{Coker} f'' \to 0$$

δ = boundary homomorphism.

(1) $\bar{u}$ injective? $\bar{u}(x) = 0 \Rightarrow u(x) = 0$

u inj $\Rightarrow x = 0$

Diagram Chasing

$y \rightsquigarrow x$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0
\end{tikzcd}$$

0

$$\begin{array}{cccccccc}
0 & \xrightarrow{} & \operatorname{Ker} f' & \xrightarrow{\bar{u}} & \operatorname{Ker} f & & \\
 & \xrightarrow{\bar{v}} & \operatorname{Ker} f'' & \rightarrow & \operatorname{Coker} f' & & \\
 & \rightarrow & \operatorname{Coker} f & \rightarrow & \operatorname{Coker} f'' & \rightarrow & 0
\end{array}$$

(2) $\operatorname{Ker}(\bar{v}) = \operatorname{Im}(\bar{u})$. $\bar{v}(\bar{u}(x)) = v(u(x)) = 0$.

Show $\operatorname{Ker}(\bar{v}) \subset \operatorname{Im}(\bar{u})$: Take $x \in \operatorname{Ker} f$,

then $f(x) = 0$, $\bar{v}(x) = 0 \Rightarrow v(x) = 0$. $\Rightarrow \exists\, y \in M'$

s.t. $u(y) = x$.

We need: $y \in \operatorname{Ker}(f')$: Commutativity implies

$f \circ u(y) = 0 = u' \circ f'(y)$. u' inj $\Rightarrow f'(y) = 0$ $\blacksquare$

Diagram Chasing

$$\begin{tikzcd} & & y & x & \\ 0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0 \end{tikzcd}$$

$$\begin{array}{llllll} 0 & \to & \operatorname{Ker} f' & \to & \operatorname{Ker} f & \\ & \to & \operatorname{Ker} f'' & \xrightarrow{\delta} & \operatorname{Coker} f' & \\ & \to & \operatorname{Coker} f & \to & \operatorname{Coker} f'' & \to \quad 0 \end{array}$$

Boundary homomorphism?

Take $x \in \operatorname{Ker} f''$. $f''(x) = 0$. $\exists y \in M$

s.t. $v(y) = x$. $\Rightarrow f(y) \in \ker(v')$. $\Rightarrow f(y) \in \operatorname{Im}(u')$,

(u' injective) $\Rightarrow \exists!\, z$ s.t. $u'(z) = f(y)$.

Take $\delta(x) = \bar{z}$ (Image of z in Coker f).

Additive Functions & Exact Sequences

Defn Let $\lambda : C \to \mathbb{Z}$ satisfying the relation that given a short exact sequence

$$0 \to A \to B \to C \to 0$$

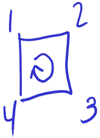
$\lambda(A) + \lambda(C) = \lambda(B)$. Then λ is called an additive function.

An Example from Topology

[Credit to Shaun Ault on Math StackExchange for the example.]

Let V = vertices, E = edges, F = faces

$$\mathbb{Z}[F] \to \mathbb{Z}[E] \to \mathbb{Z}[V]$$



$$\mapsto e_{12} + e_{23} + e_{34} - e_{14} \mapsto \cancel{v_2} - \cancel{v_1} + \cancel{v_3} - \cancel{v_2} + \cancel{v_4} - \cancel{v_3} - (\cancel{v_4} - \cancel{v_1})$$

$$0$$

$$0 \to \mathbb{Z} \to \mathbb{Z}[F] \to \mathbb{Z}[E] \to \mathbb{Z}[V] \to \mathbb{Z} \to$$

An Example from Topology

[Credit to Shaun Ault on Math StackExchange for the example.]

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}[F] \arrow[r] \arrow[d] & \mathbb{Z}[E] \arrow[r] \arrow[d] & \mathbb{Z}[V] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\lambda = & 1 & \lvert F\rvert & \lvert E\rvert & \lvert V\rvert & 1 &
\end{tikzcd}$$

$\lambda(M)$ = rank of M.

For general exact sequences

$$\sum_{n=1}^{k} (-1)^n \lambda(M_n) = 0, \qquad 1 - \lvert F\rvert + \lvert E\rvert - \lvert V\rvert + 1 = 0$$

Euler Char: $\lvert V\rvert - \lvert E\rvert + \lvert F\rvert = 2.$

Definition: Hom

Given M, N A-modules,

$$\mathrm{Hom}_A(M,N) = \{A\text{-module homomorphisms}\}$$

$\mathrm{Hom}_A(M,N)$ has structure of A-module:

- $f: M \to N$, $g: M \to N$ $\quad f-g: M \to N$
  $\quad\quad x \mapsto f(x) - g(x)$.
- $f: M \to N$, $a \in A$ $\quad af: M \to N$
  $\quad\quad x \mapsto af(x)$.

Hom$(M, \cdot)$ is a covariant functor

A-mod is the category of all A-modules.

- Objects: $N \longrightarrow \mathrm{Hom}(M,N)$.
- Morphisms: $f : N \to P$

$$\mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(M,P)$$
$$g \longmapsto f \circ g.$$

$\mathrm{Hom}(\cdot, N)$ is a contravariant functor

Functor from A-mod $\to$ A-mod.

- Objects: $M \longrightarrow \mathrm{Hom}(M, N)$.
- Morphisms: $f : M \to P$

$$\mathrm{Hom}(P, N) \longrightarrow \mathrm{Hom}(M, N)$$
$$g \longmapsto g \circ f.$$

Defn: Left-exact functor

- Covariant

If $0 \to M' \to M \to M'' \to 0$ exact

then $0 \to F(M') \to F(M) \to F(M'')$ exact.

Maps kernels to kernels.

- Contravariant

If $0 \to M' \to M \to M'' \to 0$ exact.

then $0 \to G(M'') \to G(M) \to G(M')$ exact.

maps cokernels to kernels.

Explicit computations

1. $\mathrm{Hom}(\mathbb{Z},\mathbb{Z})$.
   $\mathbb{Z}$

   $f(0)=0$. $f(1)=n$.

   $\Rightarrow f(k)=kf(1)=kn$.

   $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\simeq\mathbb{Z}$.

   $\mathrm{Hom}_A(A,A)\simeq A$.

2. $\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Q})$.
   $\mathbb{Z}$

   $f(0)=0$. $f(1)+f(1)$

   $=2f(1)=f(2)=0$

   $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Q})\simeq 0$.

3. $\mathrm{Hom}(\mathbb{Q},\mathbb{Z})$.

   $\forall n \quad nf(\frac{1}{n})=f(1)=m \Rightarrow f(\frac{1}{n})=0 \Rightarrow f\equiv 0$.

   $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})\cong 0$.

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \cdot)$ is not right-exact

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

exact.

$$0 \to \underset{\substack{\cong \\ 0}}{\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z})} \to \overset{\substack{0 \\ \cong}}{\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z})} \xrightarrow{f} \underset{\mathbb{Z}_n}{\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)} \xrightarrow{g} 0$$

$\ker(g) = \mathbb{Z}_n$

$\mathrm{im}(f) = 0.$

$\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ is not right-exact

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

exact

$$0 \to \underset{\overset{\text{\tiny |||}}{0}}{\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})} \xrightarrow{f} \underset{\mathbb{Z}}{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})} \xrightarrow{g} \overset{\overset{0}{\text{\tiny |||}}}{\mathrm{Hom}(\mathbb{Q}, \mathbb{Z})} \to 0$$

$\mathrm{im}\, f = 0$, $\ker(g) = \mathbb{Z}$ $\Rightarrow$ not right exact