ECCO 2012: Positive Grassmannian. Exercises Lecture 2

1. Calculate the number of matroids of rank $k = 2$ on $n = 4$ elements. (List all matroid polytopes.)

2. Calculate the number of positroids of rank $k = 2$ on $n = 4$ elements and check that it equals the number of decorated permutations on 4 elements with two left arcs.

3. Cut a decorated permutation between the points $i - 1$ and i. Show that the number of left arcs is the same for all is.

4. Define the Eulerian numbers $A_{k,n}$ by the formula:

$$\sum_{i=1}^{\infty} i^n x^i = \frac{\sum_{k=0}^{\infty} A_{k,n} x^k}{(1-x)^{n+1}}.$$

The left hand side $f_n(x)$ satisfies the equation $f_n(x) = x f'_{n-1}(x)$. Deduce the recurrence relation:

$$A_{k,n} = (n - k + 1) A_{k-1,n-1} + k A_{k,n-1}.$$

(see Figure 1 for an illustration of the recurrence in Euler's triangle)

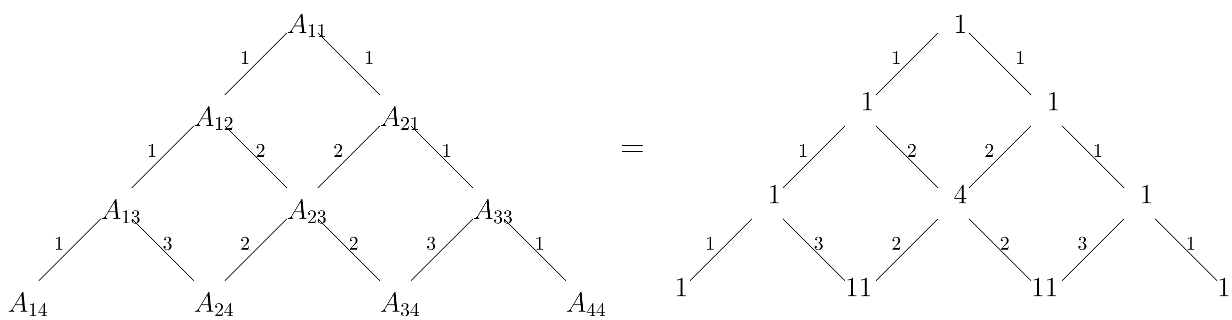

Figure 1: Euler's triangle

5. Show that $A_{k,n}$ (from the previous problem) equals the number of permutations π of size n with $k - 1$ descents (i.e., the indices i such that $\pi_i > \pi_{i+1}$). (One way to solve the problem is to check Euler's triangle recurrence for the number of permutations with a given number of descents.)

6. Show that $A_{k,n}$ equals the number of permutations π of size n with k exceedances (i.e., the indices i such that $\pi_i \geq i$). (You can construct a bijection that transforms descents into exceedances, or check Euler's triangle recurrence.)