ECCO 2012: Positive Grassmannian. Exercises Lecture 4

1. Show that for a perfect orientation of a plabic graph, the number k of boundary edges directed inside the disk satisfies

$$k - (n - k) = \#\{\text{black vertices}\} - \#\{\text{white vertices}\}.$$

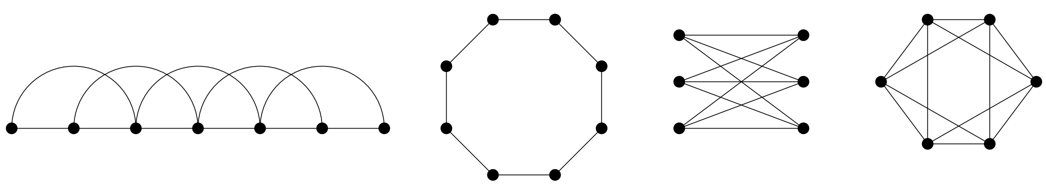
2. Recall that for a plabic graph G we define $\mathcal{M}_G = \{I_P \mid P \text{ perfect orientation }\}$ where I_P is the set of boundary vertices with edges directed in P to the interior of the disk. Prove that for a plabic graph G, $\mathcal{M}_G$ is a matroid. (Check the exchange axiom.)

3.

(a) Show that local moves of a plabic graph do not change the matroid $\mathcal{M}_G$.

(b) Show that local moves of a plabic graph do not change the decorated permutation associated with the graph.

4. Calculate chromatic polyonomials and numbers of acyclic orientations of the following graphs:



(a) (b) (c) (d)

5. Show that the number A_G at acyclic orientations of G satisfies the deletion-contraction recurrence:

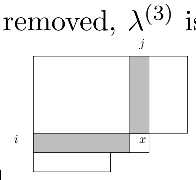
$$A_G = A_{G\backslash e} + A_{G/e}.$$

6.

(a) Check that polynomials $F_\lambda(q)$ and chromatic polynomials $\chi_\lambda(q)$ satisfy the recurrences

$$F_\lambda = qF_{\lambda^{(1)}} + F_{\lambda^{(2)}} + F_{\lambda^{(3)}} - F_{\lambda^{(4)}}$$
$$q\chi_\lambda = q\chi_{\lambda^{(1)}} - \chi_{\lambda^{(2)}} - \chi_{\lambda^{(3)}} + \chi_{\lambda^{(4)}},$$

where $\lambda^{(1)}$ be the shape λ with a corner box x at (i, j) removed, $\lambda^{(2)}$ is the shape λ with ith row removed, $\lambda^{(3)}$ is λ with jth column removed, $\lambda^{(4)}$ is λ with ith row and jth column removed



.

(b) Deduce that the number of hook diagrams of shape λ equals the number of acyclic orientations as the bipartite graph G_λ.

(c) ($\star\star$) Find a bijective proof for (b).