Notes by Zvi Rosen. Thanks to Alyssa Palfreyman for supplements.

1. Tuesday, June 19, 2012

1.1. Squarefree Monomial Ideals.

Definition 1.1. A *simplicial complex* Δ on $\{1,2,\ldots,n\}$ is a collection of subsets such that $\sigma \in \Delta$ and $\tau \subset \sigma \Rightarrow \tau \in \Delta$.

Example 1.2 (n = 5)**.** Δ = all subsets of $\{1,2,3\},\{2,4\},\{3,4\},\{5\}$. (See Figure 1.) The vector $f = (1,5,5,1)$ indicates the number of sets in the simplicial complex with the given cardinality.

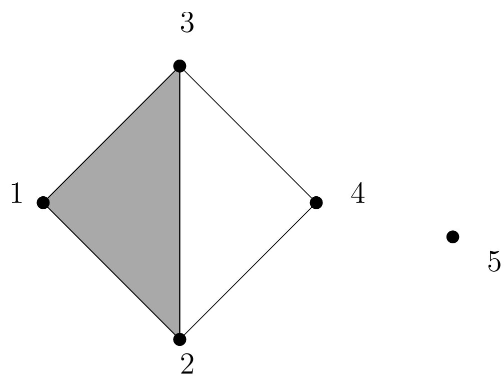

Figure 1. Simplicial Complex.

Definition 1.3. The *Stanley-Reisner ideal* of Δ is the monomial ideal
$$I_\Delta = \langle x^\tau : \tau \notin \Delta \rangle.$$

Remark 1.4. We identify subsets $\tau \subset \{1,2,\ldots,n\}$, vectors in $\{0,1\}^n$ and squarefree monomials $x^\tau = \prod_{i\in\tau} x_i$.

Example 1.5. From the simplical complex Δ above, we have
$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_3x_4, x_2x_5, x_3x_5, x_4x_5\rangle.$$

Theorem 1.6. *The map $\Delta \to I_\Delta$ is a bijection between simplicial complexes on $\{1,2,\ldots,n\}$ and square free monomial ideals in $S = K[x_1,\ldots,x_n]$. Furthermore,*
$$I_\Delta = \bigcap_{\sigma\in\Delta} \langle x_i : i \notin \sigma\rangle.$$

The facets (minimal non-faces) suffice to generate the ideal.

Example 1.7. Again, from the above, we have
$$I_\Delta = \langle x_4, x_5\rangle \cap \langle x_1, x_2, x_5\rangle \cap \langle x_1, x_3, x_5\rangle \cap \langle x_1, x_2, x_3, x_4\rangle.$$

Definition 1.8. The *Alexander dual* Δ^* consists of the complements of the non-faces of Δ.

Example 1.9. We can construct $I_{\Delta*}$ from the monomial generators of I_Δ or from the primary decomposition:
$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_3x_4, x_2x_5, x_3x_5, x_4x_5\rangle$$
$$\Rightarrow I_{\Delta^*} = \langle x_1, x_4\rangle \cap \langle x_1, x_5\rangle \cap \langle x_2, x_3, x_4\rangle \cap \langle x_2, x_5\rangle \cap \langle x_3, x_5\rangle \cap \langle x_4, x_5\rangle.$$

$$I_\Delta = \langle x_4, x_5\rangle \cap \langle x_1, x_2, x_5\rangle \cap \langle x_1, x_3, x_5\rangle \cap \langle x_1, x_2, x_3, x_4\rangle$$
$$\Rightarrow I_{\Delta^*} = \langle x_4x_5, x_1x_2x_5, x_1x_3x_5, x_1x_2x_3x_4\rangle.$$

1.2. **Hilbert Series ("Inclusion-Exclusion").**

Definition 1.10. An S- module M is $\mathbb{N}^n$*-graded* if $M = \bigoplus_{b\in\mathbb{N}^n} M_b$ and $x^a M_b \subseteq M_{a+b}$. Its *Hilbert Series* is:

$$H(M,\bar{x}) = \sum_{a\in\mathbb{N}^n} \dim_K(M_a)\cdot x^a.$$

Example 1.11.

$$H(S,x) = \prod_{i=1}^{n} \frac{1}{1-x_i} = \text{ sum of all monomials in } S.$$

If I is a monomial ideal, then $H(S/I,X) =$ sum of all monomials not in I.

Definition 1.12. The K-polynomial of M is the numerator of

$$H(M,x) = \frac{K(M,x)}{\prod_{i=1}^{n}(1-x_i)}.$$

Theorem 1.13. *The Stanley-Reisner ring has:*

$$K(S/I_\Delta,x) = \sum_{\sigma\in\Delta}\left(\prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j)\right).$$

Example 1.14. For the square graph $abcd$:

$$I_\Delta = \langle ac, bd\rangle.$$

$$K = 1 - ac - bd + abcd.$$

We can use the theorem to calculate:

$$K = (1-a)(1-b)(1-c)(1-d) + a(1-b)(1-c)(1-d) + \cdots + a(1-b)(1-c)d.$$

Corollary 1.15 (Stanley's Green Book)**.** *If* $d = \dim(\Delta)+1$ *then*

$$H(S/I_\Delta; t,t,\ldots,t) = \frac{1}{(1-t)^n}\sum_{i=0}^{d} f_{i-1}t^i(1-t)^{n-i}$$

$$= \frac{1}{(1-t)^d}\sum_{i=0}^{d} f_{i-1}t^i(1-t)^{d-i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

1.3. **Monomial Matrices.** Consider a sequence of $\mathbb{N}^n$-graded S-modules

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_{e-1}} F_{e-1} \xleftarrow{\phi_e} F_e \leftarrow 0.$$

Each ϕ_i preserves $\mathbb{N}^n$ grading: it can be written as a matrix with entries in K and row/column labels in $\mathbb{N}^n$.

Definition 1.16.
- $\mathcal{F}_\bullet$ is a *complex* if $\phi_i \circ \phi_{i+1} = 0 \forall i$.
- $\mathcal{F}_\bullet$ is *exact* in homological degree i if $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$.
- $\mathcal{F}_\bullet$ is a *free resolution* of M if it is exact everywhere except in homological degree 0, where $M = F_0/\mathrm{im}(\phi_1)$.

Theorem 1.17 (Hilbert's Syzygy Theorem)**.** *There exists such a free resolution of length* $\leq n$ *(the length of the grading).*

Remark 1.18. We use this theorem to compute K-polynomials.

Example 1.19.

$$K = 1 - ac - bd + abcd.$$

$$0 \leftarrow S_{0000} \xleftarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} S^2_{\substack{1010\\0101}} \xleftarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} S_{1111} \leftarrow 0.$$

Also, could be read as:

$$0 \leftarrow S_{0000} \xleftarrow{\begin{bmatrix} ac & bd \end{bmatrix}} S^2_{\substack{1010\\0101}} \xleftarrow{\begin{bmatrix} bd \\ -ac \end{bmatrix}} S_{1111} \leftarrow 0.$$

1.4. **Betti Numbers.**

Definition 1.20. If $\mathcal{F}_\bullet$ is a minimal free resolution of M and $F_i = \bigoplus_{a \in \mathbb{N}^n} (S_a)^{\beta_{i,a}}$, then the *$i$-th Betti number of M* in degree a is $\beta_{i,a} = \beta_{i,a}(M)$.

Remark 1.21.

$$K(M, x) = \sum_{a \in \mathbb{N}} \sum_{i=0}^{l} (-1)^i \beta_{i,a}(M) x^a.$$

Definition 1.22. For a monomial ideal I and degree $b \in \mathbb{N}^n$, define the *Koszul simplicial complex*

$$\mathcal{K}^b(I) = \{\tau \,|\, x^{b-\tau} \in I\}.$$

Theorem 1.23 (Hochster)**.** *The Betti numbers of I and S/I in degree i can be expressed as*

$$\beta_{i,b}(I) = \beta_{i+1,b}(S/I) = \dim_K \tilde{H}_{i-1}(\mathcal{K}^b(I); K).$$

(Here, we are discussing reduced homology.)

Exercise 1.24. Calculate the Alexander dual. See Figure 2.

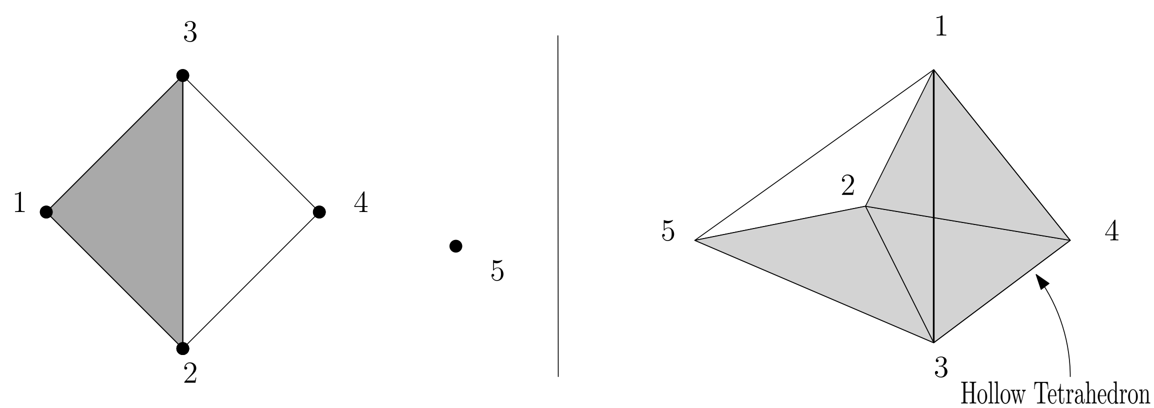

Figure 2. Simplicial Complex and its Alexander Dual.

1.5. **Questions.**

Question 1.25. How can you tell if a complex is a resolution, i.e. exact at a given step?

Use Macaulay 2. Try to check dimension, and Gröbner bases are your friend.

Question 1.26. How does one check minimality of a free resolution?

No nonzero constants in the matrix output by Macaulay2.

2. Wednesday, June 20, 2012

2.1. **Borel-fixed Monomial Ideals.** Let $S = K[x_1, \dots, x_n]$ be a ring with $\mathbb{N}$ grading, such that $\mathrm{char}(K) = 0$.

Let $GL_n(K) = \{$invertible $n\times n$ matrices$\}$, called the general linear group. Let $B_n(K) = \{$upper-triangular $n \times n$ matrices$\}$, called the Borel group. Let $T_n(K) = \{$diagonal $n \times n$ matrices$\}$, called the Torus group.

$$T_n(K) \subset B_n(K) \subset GL_n(K).$$

Proposition 2.1. *An ideal $I \subset S$ is fixed under T_n iff I is a monomial ideal.*

Proof by Example. Consider $f = 11x^2y + 17yz + 19xz^3 \in I \subset K[x, y, z]$, a torus-fixed ideal. Scale x, y, z by $2, 3, \dots$ (for example).

$$\begin{pmatrix} 11 & & \\ & 17 & \\ & & 19 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2^3 & 2^2 & 2^4 \\ 3^3 & 3^2 & 3^4 \end{pmatrix} \begin{pmatrix} x^2y \\ yz \\ xz^3 \end{pmatrix} \in \begin{pmatrix} I \\ I \\ I \end{pmatrix}$$

By fudging with the scaling numbers, you can get each entry to stand on its own. □

Proposition 2.2. *I is GL_n-fixed iff $I = \langle x_1, \dots, x_n\rangle^d$ for some $d \in \mathbb{N}$.*

Proposition 2.3. *For a monomial ideal I, the following are equivalent:*

(1) I is Borel-fixed.
(2) If $m \in I$ is divisible by x_j, then $m\frac{x_i}{x_j} \in I$, for $i < j$.

Fix a term order $<$ on S. If I is any ideal in S, then its generic initial ideal is

$$gin_<(I) := in_<(g \circ I).$$

where g is a random matrix in $GL_n(K)$, i.e. in a suitable Zariski open subset.

Theorem 2.4 (Theorem 15.20 in Eisenbud)**.** *$gin_<(I)$ is Borel-fixed.*

2.2. **Gröbner Basis Review.**

Example 2.5. Consider $I = \langle \Lambda^+_{(3)} \rangle$, under the lex order $x > y > z$. By definition,

$$I = \langle x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3\rangle.$$

The Gröbner basis is:

$$GB = \{x + y + z, y^2 + yz + z^2, z^3\}.$$

$$in_<(I) = \langle x, y^2, z^3\rangle.$$

$$S/I \cong_K K\{1, y, z, yz, z^2, yz^2\}.$$

Hilbert function ($\mathbb{N}$-grading):

d	0	1	2	3	4	5	6
S	1	3	6	10	15	21	28
S/I	1	2	2	1	0	0	0
I	0	1	4	9	15	21	28

$$gin_<(I) = \langle x, y^2, yz^2, z^4 \rangle.$$

Example 2.6 (**GB for Submodules**)**.**

$$M = \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x+y \\ x+y \end{bmatrix} \right\rangle \subset S^2$$

TOP (term over position) Gröbner basis rules ties in favor of the term with greater weight. In our case,

$$GB = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix} \right\} \qquad in_<(M) = \langle xe_1, xe_2 \rangle.$$

$$GB = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 - y^2 \end{bmatrix} \right\} \qquad in_<(M) = \langle xe_1, ye_1, (x^2-y^2)e_2 \rangle.$$

2.3. **Eliahou-Kervaire Resolution.**

Lemma 2.7. *Each monomial m in Borel-fixed monomial ideal*

$$I = \langle m_1, m_2, \ldots, m_r \rangle.$$

can be written uniquely as a product $m = m_i \cdot m'$, with $\max(m_i) \le \min(m')$. Let $u_i = \max(m_i)$.

Proposition 2.8. *The $\mathcal{K}$-polynomial of S/I equals*

$$\mathcal{K}(S/I, x) = 1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{u_i - 1} (1 - x_j).$$

Example 2.9.

$$I = \langle x_1^2, x_1x_2, x_2^3, x_1x_3^3 \rangle.$$

$$\begin{aligned} \Rightarrow \mathcal{K} &= 1 - x_1^2 - x_1x_2(1-x_1) - x_2^3(1-x_1) - x_1x_3^3(1-x_1) - x_1x_3^3(1-x_1)(1-x_2) \\ &= 1 - x_1^2 - x_1x_2 - x_2^3 - x_1x_3^3 + x_1^2x_3^2 + x_1x_2x_3^3 + x_1x_2^3 + x_1^2x_2 - x_1^2x_2x_3^3. \end{aligned}$$

This suggests the minimal free resolution

$$0 \longleftarrow S \longleftarrow S^4 \longleftarrow S^4 \longleftarrow S \longleftarrow 0.$$

Theorem 2.10. *Let $M \subset S^r$ be the module of first syzygies on a Borel-fixed monomial ideal I. Then M has a POT Gröbner basis whose initial module $in(M)$ has a linear free resolution. Moreover, $S^r/in(M)$ and $I \cong S^r/M$ have the same Betti numbers, namely:*

$$\beta_i = \sum_{j=1}^{r} \binom{u_j - 1}{i}.$$

Example 2.11 (Example 2.19 from Sturmfels-Miller)**.**

$$I = \langle x_1x_2x_4^4, x_1x_2x_3x_4^2, x_1x_3^6, x_1x_2x_3^2, x_2^6, x_1x_2^2, x_1^2 \rangle.$$

See text or Macaulay2 Code for more detail.

2.4. **Lex-Segment Ideals.** Fix a Hilbert function $H : \mathbb{N} \to \mathbb{N}$ of some homogeneous ideal $I \subset S$.
Let L_d be the K-span of the $H(d)$ largest monomials in the lex order on S_d and define

$$L = \bigoplus_{d=0}^{\infty} L_d.$$

Proposition 2.12 (Macaulay 1927)**.** *The graded vector space L is a Borel-fixed ideal.*

Theorem 2.13 (Macaulay's Theorem)**.** *For every $d \in \mathbb{N}$, the lex-segment ideal L has at least as many generators as every other (monomial) ideal with the same Hilbert function H.*

Example 2.14. Intersect of a quadric and cubic surface in $\mathbb{P}^3$. As an ideal, this is generated by a quadratic and cubic homogeneous polynomial in 4 variables.

The Hilbert Series is:

$$\frac{1 - t^2 - t^3 + t^5}{(1-t)^4} = \frac{1 + 2t + 2t^2 + t^3}{(1-t)^2} = 1 + 4t + \sum_{r=2}^{\infty}(6r-3)t^r.$$

$$gin_{revlex}(I) = \langle x_1^2, x_1x_2^2, x_2^4\rangle \qquad gin_{lex}(I) = \langle x_1x_3^6, x_2^6, x_1x_2x_4, x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2\rangle.$$

The lex-segment ideal L has 18 generators, more than from any other term ordering.

2.5. **Question Session.**

Question 2.15. What is a syzygy?

Syzygy, historically from astronomy, means relations. So, syzygies are relations among the generators of an ideal.

Question 2.16. Describe a Gröbner Basis for submodules.

An element of S^n is a sum of monomials times coordinate vectors: $\sum \gamma_i m_i e_{j_i} \in S^n$. Given these elements, you are setting a term order in each coordinate. Similar to ordering of variables, we also want the position to take a certain priority in the variable ordering.

In POT, we have $m_1e_i > m_2e_j$ iff $i > j$ or $i = j$ and $m_1 > m_2$.

In TOP, we have $m_1e_i > m_2e_j$ iff $m_1 > m_2$ or $m_1 = m_2$ and $i > j$.

Question 2.17. How do we find the decomposition $m = m_i \cdot m'$?

Since the ideal is Borel-fixed, we can trade higher variables for lower variables while staying in the generating set.

3. Thursday, June 21, 2012

3.1. **Staircases.** First, let us look at a 2-dimensional case.

Example 3.1. Consider the ideal $I \subset k[x,y]$ as follows.

$$I = \langle x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\rangle.$$

such that $a_1 > a_2 > \cdots > a_r$, and $b_1 < b_2 < \cdots < b_r$.

This can be portrayed in a staircase diagram as in Figure 3.

Let us consider the $\mathcal{K}$ polynomial of this ideal:

$$\mathcal{K}(S/I, x, y) = (1-x)(1-y)\sum\{x^iy^j \notin I\} = 1 - \sum_{i=1}^{r} x^{a_i}y^{b_i} + \sum_{j=1}^{r-1} x^{a_j}y^{b_{j+1}}.$$

The first sum corresponds to the inner corners, and the second sum to the outer corners.

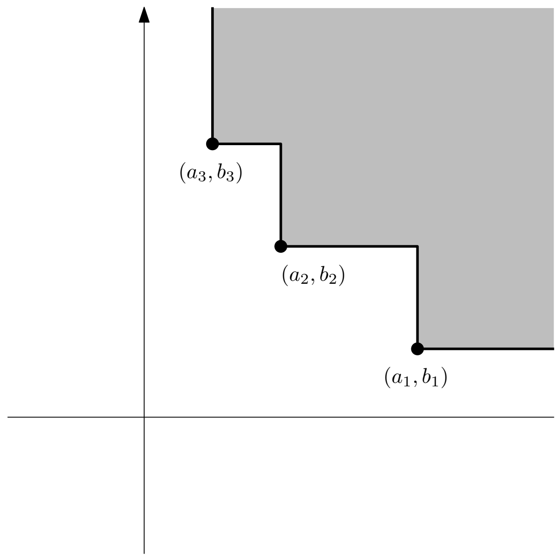

FIGURE 3. Staircase Diagram.

Proposition 3.2. *The minimal free resolution of S/I equals*

$$0 \longleftarrow S \longleftarrow S^r \longleftarrow S^{r-1} \longleftarrow 0.$$

The minimal first syzygies are $(y^{b_{i+1}-b_i}e_i - x^{a_i - a_{i+1}}e_{i+1})$.

Proposition 3.3 (Irreducible Decomposition)**.**

$$I = \langle y^{b_1}\rangle \cap \langle x^{a_1}, y^{b_2}\rangle \cap \langle x^{a_2}, y^{b_3}\rangle \cap \cdots \cap \langle x^{a_r-1}, y^{b_r}\rangle \cap \langle x^{a_r}\rangle,$$

where the first (resp. last) intersectand is deleted if $b_1 = 0$ *or* $a_1 = 0$.

Definition 3.4. The *Buchberger graph* $Buch(I)$ of a monomial ideal $I = \langle m_1, m_2, \ldots, m_r\rangle$ has:

- Vertices $1, 2, \ldots, r$,
- Edges $\{i, j\}$ for i, j such that there is no k for which $m_k | lcm(m_i, m_j)$ and m_k has smaller degree in every variable occurring in $\mathrm{lcm}(m_i, m_j)$.

Proposition 3.5. *The module of syzygies on I is generated by the syzygies*

$$\sigma_{ij} = \frac{lcm(m_i, m_j)}{m_i}e_i - \frac{lcm(m_i, m_j)}{m_j}e_j$$

corresponding to edges in the Buchberger graph.

See Figure 3.2 in Sturmfels-Miller for an example. The ideal's decomposition can be read by breaking the cubical complex into cuboids. In labeling the Buchberger graph, in each face, you write the vector of the least common multiple of the vertices. The $\mathcal{K}$ polynomial is 1 − (vertex labels) + (edge labels) − (face labels).

In general, the Buchberger graph is not planar. But it has nice properties under genericity conditions.

3.2. **Genericity and Deformations.**

Definition 3.6. A monomial ideal $I \subset K[X, Y, Z]$ is *strongly generic* if any generators $x^i y^j z^k$ and $x^{i'} y^{j'} z^{k'}$ have the property that $i \neq i'$ unless they are both zero, and similarly for the other indices.

Proposition 3.7. *If I is strongly generic, then the Buchberger graph is planar and connected. If I is also Artinian, then $Buch(I)$ consists of the edges of a triangulated triangle (thus, 3-connected).*

Definition 3.8. A planar map is a graph together with an embedding into a surface homeomorphic to $\mathbb{R}^2$.

Theorem 3.9. *Given a strongly generic monomial ideal I in $K[X,Y,Z]$, the planar map $Buch(I)$ provides a minimal free resolution of I:*

$$0 \longleftarrow S \longleftarrow S^r \xleftarrow{\partial_E} S^e \xleftarrow{\partial_F} S^f \longleftarrow 0.$$

The differentials in this sequence are:

$$\partial_E(e_{ij}) = \frac{m_{ij}}{m_j}e_i - \frac{m_{ij}}{m_i}e_j.$$

$$\partial_F(e_R) = \sum_{\text{edges } \{i,j\}\subset R} \pm\frac{m_R}{m_{ij}}e_{ij}, \text{ where } m_R = lcm(m_i|i \in R).$$

Question 3.10. What if I is not strongly generic?

Introduce a polynomial ring $S_\varepsilon = K[x^\varepsilon, y^\varepsilon, z^\varepsilon]$ for $\varepsilon = \frac{1}{N}$ for some $N \in \mathbb{N}$ which contains $S = K[X,Y,Z]$.

Consider monomial ideals:

$$I = \langle m_1, m_2, \ldots, m_r\rangle \subset S, \qquad \text{and } I = \langle m_{\varepsilon,1}, m_{\varepsilon,2}, \ldots, m_{\varepsilon,r}\rangle \subset S_\varepsilon.$$

Say I_ε is a strong deformation of I if the partial order on $\{1,2,\ldots,r\}$ by x-degree of the m_ε refines the partial order of the m_i and same for y and z.

Example 3.11.

$$I = \langle X, Y, Z\rangle^3.$$

We could approach this problem using Borel-fixed ideal theory, Eliahou-Kervaire resolutions, or even tropical strategy. We will use the Buchberger graph of I and I_ε. (See Figure Δ).

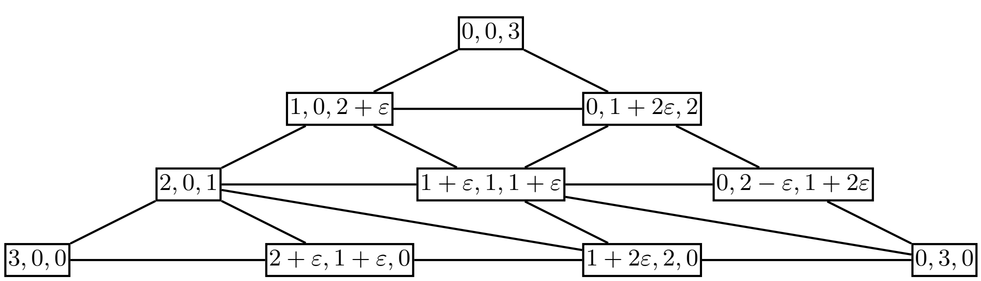

FIGURE 4. $Buch(I_\varepsilon)$ after deformation.

Proposition 3.12. *Specializing the labels of the vertices, edges, and faces of the planar Buchberger graph $Buch(I_\varepsilon)$ under $\varepsilon = 0$ yields a planar map resolution of I. (usually not minimal).*

Question 3.13. Can you always make it minimal?

Yes. See Section 3.5 of Sturmfels-Miller.

Corollary 3.14. *Let r be the number of generators of an ideal, e the number of first syzygies, and f the number of second syzygies. Then, $e \le 3r - 6$ and $f \le 2r - 5$.*

4. Friday, June 22, 2012

Definition 4.1. A *polyhedral complex* in $\mathbb{R}^m$ is a finite set X of convex polytopes such that

- If $P \in X$ and $F \subset P$ is a face, then $F \in X$.
- If $P, Q \in X$, then $P \cap Q$ is a face of both P and Q.

X has a (reduced) chain complex (over $\mathbb{Z}$) with boundary maps

$$\partial(F) = \sum_{\substack{\text{facets}\\ G \subset F}} \mathrm{sign}(G, F) \cdot G.$$

Definition 4.2. X is a *labelled cell complex* if its r vertices are labelled by vectors $a_1, \ldots, a_r \in \mathbb{N}^n$. The label of any face $F \in X$ is given by

$$x^{a_F} = \mathrm{lcm}(x^{a_i} | i \in F).$$

The monomial matrix on X uses this chain complex for scalar entries with row and column labels a_F, for $F \in X$.

The *cellular free complex* $\mathcal{F}_X$ is the resulting complex of $\mathbb{N}^n$ graded free S-modules.

$$\mathcal{F}_X = \bigoplus_{\mathcal{F} \in X} S(-a_F). \qquad \partial(F) = \sum_{\substack{\text{facets}\\ G \subset F}} \mathrm{sign}(G, F) \cdot x^{a_F - a_G} \cdot G.$$

We call $\mathcal{F}_X$ a *cellular resolution* if it is exact.

Example 4.3. Consider the octahedron cell complex as in Figure 5. By counting the faces of various dimension, we obtain the cellular free complex below:

$$0 \leftarrow S \leftarrow S^6 \leftarrow S^{12} \leftarrow S^8 \leftarrow S \leftarrow 0.$$

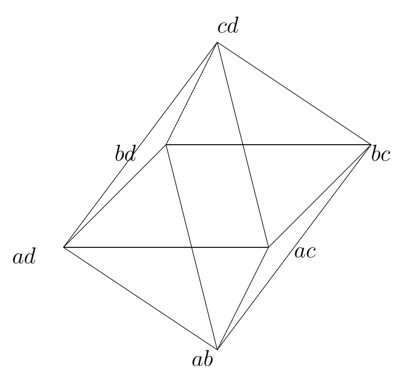

Figure 5. Labelled Cell Complex.

If Q is an order ideal in $\mathbb{N}^n$, then $X_Q = \{F \in X | a_F \in Q\}$ is a labeled sub complex of X.

Example 4.4. $X_{\preceq \mathbf{b}}$ and $X_{\prec \mathbf{b}}$.

Proposition 4.5. *$\mathcal{F}_X$ is a cellular resolution iff the cell complex $X_{\preceq \mathbf{b}}$ is acyclic over K for all $\mathbf{b} \in \mathbb{N}^n$.*

In this case, $\mathcal{F}_X$ resolves S/I where $I = \langle x^{a_v} | v \text{ vertex} \in F \rangle$.

Example 4.6 (Example 4.3 Continued)**.** Take $X_{\preceq abc}$, where X is the labeled cell complex from Example 4.3. The resulting subcomplex as depicted in Figure 6 is acyclic, as are all other subcomplexes. Therefore, $\mathcal{F}_X$ is a cellular resolution. However, it is not minimal, since the edge labels match the face label.

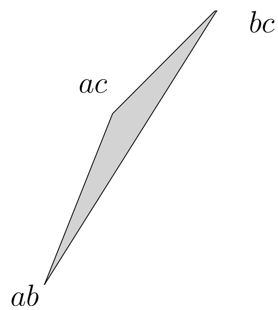

FIGURE 6. Subcomplex $X_{\preceq abc}$.

Theorem 4.7. *Write $\widetilde{H}_i(X, k)$ for the reduced homology. If $\mathcal{F}_X$ resolves I then*

$$\beta_{i,b}(I) = \dim_k \widetilde{H}_{i-1}(X_{\prec b}, k).$$

Example 4.8 (Example 4.6 Continued)**.** To calculate $\beta_{2,abcd}(I)$, we look at the subcomplex $X_{\prec abcd}$, depicted in Figure 7. The $\mathcal{K}$-polynomial of this ideal is:

$$\mathcal{K} = 1 - ab - ac - ad - bc - bd - cd + abc + abd + acd + bcd - 3abcd.$$

Because $\dim_k \widetilde{H}_1(X_{\prec b}, k) = 3$, we have $\beta_{2,abcd}(I) = 3$.

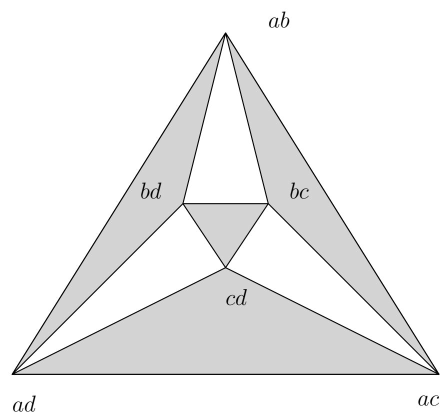

FIGURE 7. Subcomplex $X_{\prec abcd}$.

Theorem 4.9. *If $\mathcal{F}_X$ is a cellular resolution of I, then the $\mathcal{K}$-polynomial of I is the $\mathbb{N}^n$-graded Euler characteristic*

$$\mathcal{K}(S/I; X) = \sum_{F \in X} (-1)^{\dim F + 1} x^{a_F}.$$

Examples of Cellular Resolutions

(1) Planar Maps. (as we discussed in relation to the Buchberger graphs)

(2) Taylor Resolution: $X = \Delta_{r-1}$, the full $(r-1)$-simplex. This is “highly non-minimal.”

(5) Minimal Triangulation of $\mathbb{RP}^2$.

$$0 \leftarrow S \leftarrow S^{10} \leftarrow S^{15} \leftarrow S^6 \leftarrow 0$$

is exact iff $\mathrm{char}(k) \neq 2$. The corresponding cell complex has 10 vertices, 15 edges, and 6 pentagonal faces (see p. 70 of Sturmfels-Miller for a diagram.)

(3) Permutohedron Ideals.

(4) Tree Ideals. They are defined as follows:

$$I = \left\langle \left(\prod_{i \in \sigma} x_i \right)^{n-|\sigma|+1} \,\middle|\, \emptyset \subseteq \sigma \subseteq [n] \right\rangle .$$

The tree ideals are Alexander dual to permutohedron ideals. They have $(n+1)^{n-1}$ standard monomials, one for each labeled tree on $n+1$ vertices. The Hilbert Series gives the parking functions = # reduced divisors on K_{n+1}.

These objects are important for Chip Firing (see "Monomials, Binomials, and Riemann-Roch" by Manjunath and Sturmfels).

The cell complex for $n = 3$ is presented in Figure 8.

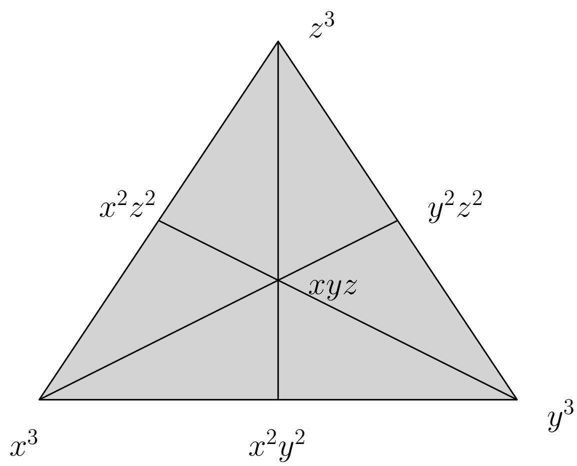

FIGURE 8. Tree ideal for $n = 3$.

(6) Simple Polytopes. Let $\mathcal{P}$ be a simple d-polytope with facets $F_1, \ldots, F_n$ and vertices $v_1, \ldots, v_r$.

Label each vertex v_i by a squarefree monomial $\prod_{v_i \notin F_j} x_j$.

The corresponding ideal $I_{\mathcal{P}}$ is the irrelevant ideal in the toric Cox ring.

Exercise 4.10. Prove that $\mathcal{F}_{\mathcal{P}}$ is a linear minimal free resolution.

Question 4.11. Does every monomial ideal have a minimal cellular resolution?

No! (M. Velasco, Journal of Algebra 2008)