Analytic Combinatorics, ECCO 2012
Exercise sheet 3

1. Show that the product of two exponential generating functions $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ is
$$A(z)B(z) = \sum_{n\geq 0} (\sum_{k\geq 0}^{n} \binom{n}{k} a_k b_{n-k}) \frac{z^n}{n!}.$$

2. (a) Let $\mathcal{P}$ be the labeled class of permutations $\mathcal{P} = SEQ(\mathcal{Z})$. We denote by $\tilde{\mathcal{P}}$ the combinatorial class obtained from $\mathcal{P}$ when we forget the labels. What are $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}_n}$?

   (b) Check that $\mathcal{P}' = SET(CYC(\mathcal{Z}))$ also describes permutations as sets of cycles, and therefore in the labeled universe $\mathcal{P} \cong \mathcal{P}'$. What is $\tilde{\mathcal{P}'}$? Show that $\tilde{\mathcal{P}} \ncong \tilde{\mathcal{P}}'$.

   (c) In the unlabeled universe, what is the relationship between $SEQ_{\geq 1}(\mathcal{Z})$, $MSET_{\geq 1}(\mathcal{Z})$, $CYC(\mathcal{Z})$?

   (d) Show that in the labeled universe the identity $SET \circ CYC \cong SEQ$ is true.

3. (a) Find $R_n^{(2)}$, $R_n^{(3)}$ and $R_n^{(n)}$.

   (b) Find $S_n^{(2)}$, $S_n^{(3)}$ and $S_n^{(n)}$.

   (c) Show that $R_n = \frac{1}{2} \sum_{l\geq 0} \frac{l^n}{2^l}$ and $S_n = \frac{1}{e} \sum_{l\geq 0} \frac{l^n}{l!}$.

   (d) Find the EGF of *double surjections*, i.e. surjections where each preimage contains at least two elements.

4. (a) Let $\mathcal{W}^{(A)}$ denote the family of words over an alphabet of cardinality r, such that the number of occurrences of each letter lies in a set $A \subseteq \mathbb{N}$. Find $\mathcal{W}^{(A)}(z)$.

   (b) Find the EGF of words containing each letter at most b times, and the EGF of words containing each letter more than b times.

5. (a) Find the EGF of the class of partitions (of sets) without singletons.

   (b) Find the EGF for all permutations whose cycles have length at most r.

   (c) Find the EGF of all derangements (permutations without fixed points).

6. An alignment is a well-labeled sequence of cycles. Let $\mathcal{O}$ be the set of all alignments. Find $\mathcal{O}(z)$.