1. August 24, 2015

Algebraic topology: take "topology" and get rid of it using combinatorics and algebra.
Topological space $\mapsto$ combinatorial object $\mapsto$ algebra (a bunch of vector spaces with maps).
Applications:

(1) Dynamical Systems (Morse Theory)
(2) Data analysis. Topology can distinguish data sets from topologically distinct sets.

1.1. Euclidean topology.

Working in $\mathbb{R}^n$, the distance $d(x, y) = ||x - y||$ is a metric.

Definition 1.1. Open set U in $\mathbb{R}^n$ is a set satisfying $\forall x \in U \exists \epsilon$ s.t.

$$O_\epsilon(x) = \{y | \, ||y - x|| < \epsilon\} \subset U$$

1.2. Topological Spaces.

Definition 1.2. A topological space is a pair $(X, \mathcal{T})$ such that X is a set, and $\mathcal{T} \subseteq 2^X$ is a set of subsets of X that satisfy:

(1) $\varnothing, X \in \mathcal{T}$.
(2) If $A_1, \ldots, A_k \subset \mathcal{T}$ then $\bigcap_{i=1}^k A_i \in \mathcal{T}$. (Finite Intersection)
(3) For any collection $\{A_i\} \subseteq \mathcal{T}$, the union $\bigcup_{i \in I} A_i \in \mathcal{T}$. (Arbitrary Union)

Example 1.3. Some sample topologies:

(1) Discrete topology: $\mathcal{T} = 2^X$.
(2) Indiscrete topology: $\mathcal{T} = \{\varnothing, X\}$.
(3) The induced topology on a metric space. Metric spaces have a metric which is positive-definite, symmetric and satisfies the triangle inequality.
$\mathcal{T} = \{U \subseteq X : \forall x \in U \exists \epsilon \; s.t. \; O_\epsilon(x) \subseteq U\}$.

1.3. Topology induced by a map.

Let $(X, \mathcal{T}_X)$ be a topological space. Let $f : X \to Y$ be a map of sets. Assume $f(X) = Y$ (unclear if necessary assumption).

Then $\mathcal{T}_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}$ is a topology. Notation: $\mathcal{T}_Y = f_*(\mathcal{T}_X)$.

1.4. Quotient Topology.

Let $\sim$ be an equivalence relation on X. Consider $\pi : X \to X/\sim$.

Definition 1.4. Let $\pi_*(\mathcal{T}_X)$ (using the induced notation) be the quotient topology on $Y = X/\sim$.

Example 1.5. Let $X = \mathbb{R}^1$. Let $x \sim y := (x - y) \in 2\pi\mathbb{Z}$. Then $Y = (X/\sim) \cong S^1$. A map to get this would be $\pi : \mathbb{R} \to S^1, \pi(\theta) = e^{i\theta}$.

Example 1.6. $S^n = B^n/\sim$ where $x \sim y \iff ||x|| = ||y|| = 1$. Think about folding a disk of aluminum foil over a 2-sphere, so that the edges all go to the north pole.

Definition 1.7. A map of topological spaces $f : X \to Y$ is continuous iff for all open $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

2. August 26, 2015

2.1. Review.

- Topological space $(X, \mathcal{T}_X)$
- Forgotten definition: Closed set is the complement of an open set.
- Induced topologies
  - by a map $f : X \to Y$.
  - by a metric (X, d_X).
  - by a subset $A \subset X$. $(X, \mathcal{T}_X) \to (A, \mathcal{T}_A)$. $\mathcal{T}_A = \{A \cap U$ where $U \in \mathcal{T}_X\}$.
- Continuous maps are maps where the preimage of an open set is open.

2.2. **Homeomorphism.**

Definition 2.1. Let $f : X \to Y$ be a map of spaces; f is a homeomorphism if 1) f is a bijection, 2) f is continuous, and 3) f^{-1} is continuous.

Remark 2.2. If f is a bijection AND $f(x)$ is continuous, f^{-1} is not necessarily continuous. For example, if f is the identity, and $\mathcal{T}_1$ is discrete and $\mathcal{T}_2$ is indiscrete.

Note that $X \cong Y$ is an equivalence relation. So the category of topological spaces is often defined modulo homeomorphism.

Example 2.3. The two realizations of S^n that we defined last class are homeomorphic.

Example 2.4. The open interval is homeomorphic to $\mathbb{R}^1$ under the tangent function.

Example 2.5. The open interval and the half-open interval (using the induced topology) are not homeomorphic.

2.3. **Connected spaces.** To prove this last example, we make two definitions:

Definition 2.6. A space X is *connected* if the only subsets of X that are both open and closed are X and $\varnothing$.

A space X is *disconnected* if $\exists U, V$ nonempty open s.t. $X = U \cup V$ and $U \cap V = \varnothing$.

A space is connected if and only if it is not disconnected.

Proof. Let $X = (0, 1)$ and $Y = (0, 1]$, $f : X \to Y$. Take $x = f^{-1}(1)$. Then $X \setminus x$ should be connected and open, since it is the preimage of a connected open set. However, this is not so. Why is this true? The homeomorphism acting on a disconnection will give a disconnection of the target. □

Definition 2.7. A space X is *path-connected* if given any two points $x, y \in X$ there is a continuous map $[0, 1] \to X$ with $f(0) = x$ and $f(1) = y$.

Lemma 2.8. *X path-connected implies X connected.*

The converse is not true but requires some pathological behavior.

There is an equivalence relation $\sim$ on X setting $x \sim y \iff \exists$ continuous path from x to y.

Definition 2.9. (Path-connected components of X) $:= X/\sim$.

Exercise 2.10. Let $X \cong$ the 2-sphere S^2, and Y the 2-torus T^2.

Prove these are not homeomorphic. Cut a circle out of the torus, map to the sphere. The result should be (path-)connected; however, that's impossible.

Definition 2.11. X is Hausdorff means $x \neq y \in X$ then $\exists$ open U containing x and open V containing y that are disjoint.

Example 2.12. Non-Hausdorff space: Take X and Y two copies of $\mathbb{R}^1$. Glue them together except at the origin; i.e. $X \sqcup Y/\sim$ where $\sim := x \sim y \iff x = y \neq 0$.

3. September 2, 2015

[Some classes were missed]

3.1. **Review.**

Theorem 3.1. *If M is a compact 2-dimensional manifold without boundary then:*
- *If M is orientable, $M = H(g) = \#^g \Pi^2$.*
- *If M is nonorientable, $M = M(g) = \#^g \mathbb{RP}^2$.*

Terminology: g is the genus of the surface = maximal number of closed paths one can cut out without disconnecting.

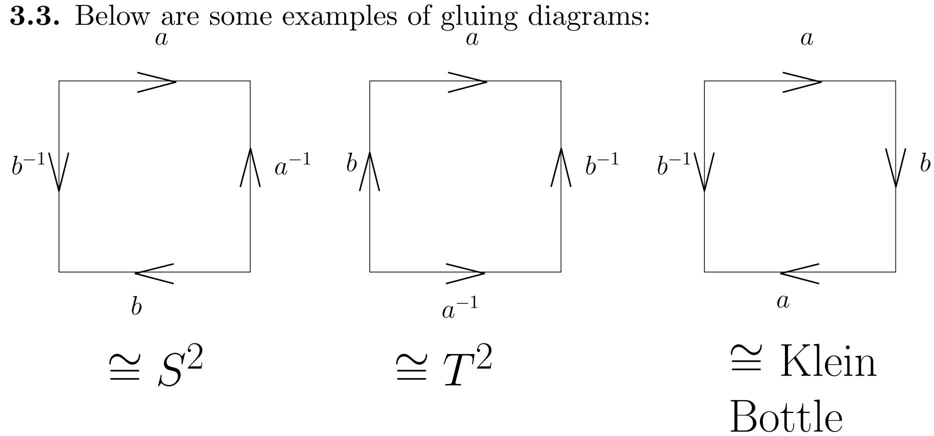
Note: No higher-dimensional analogue exists (2-dimensions is trivial). Note: $H(0) = S^2$ by definition.

3.2. **Gluing diagrams.**

Definition 3.2. Edges are "decorated" with letter:
- a means that the orientation is clockwise.
- a^{-1} means that the orientation is counterclockwise.

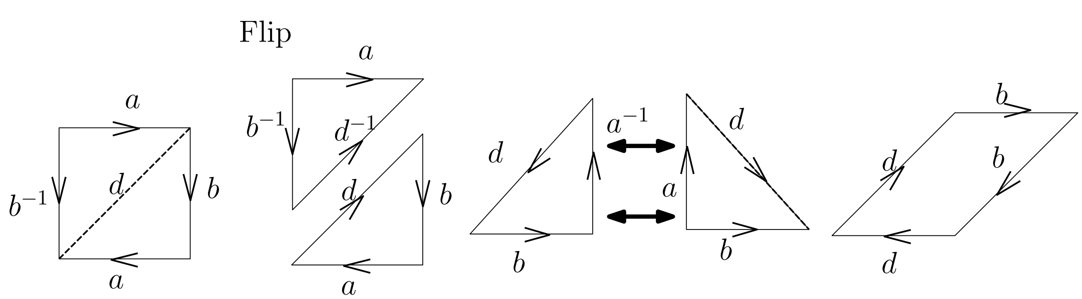
For each letter the edges are glued according to the orientation.

Example 3.3. Below are some examples of gluing diagrams:



$w = aba^{-1}cd \cdots gf$ is a word describing the circumference of a polygon. Simple properties:
(1) Cyclic permutation preserves the homeomorphism class.
(2) Inserting $aa^{-1} \cong$ connected sum with a sphere; therefore, it preserves the homeomorphism class.
(3) Concatenating two words amounts to connected sum of the corresponding manifolds (really, concatenating the inverse of one, but the inverse is isomorphic to itself).

Example 3.4. Show that the Klein bottle is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$.



Proof. $aba^{-1}b \cong abdd^{-1}a^{-1}b = (abd)(d^{-1}a^{-1}b) = (abd)(b^{-1}ad) = (daad) \cong \mathbb{RP}^2 \# \mathbb{RP}^2$. $\square$

The same logic would apply to prove that $T^2 \# \mathbb{RP}^2 \cong \#^3 \mathbb{RP}^2$; manipulating the perimeter words eventually obtains the result.

3.3. **Triangulations.** The topology of any 2-d manifold can be determined by a collection of triangles and how they are glued together.

Definition 3.5. A triangulation of a 2-d manifold M is a collection of $T_i \subset M$ s.t. if $T_i \cap T_j \neq \varnothing$ then either $T_i \cap T_j =$ one edge of each triangle or $T_i = T_j =$ a single point which is a vertex of each triangle.

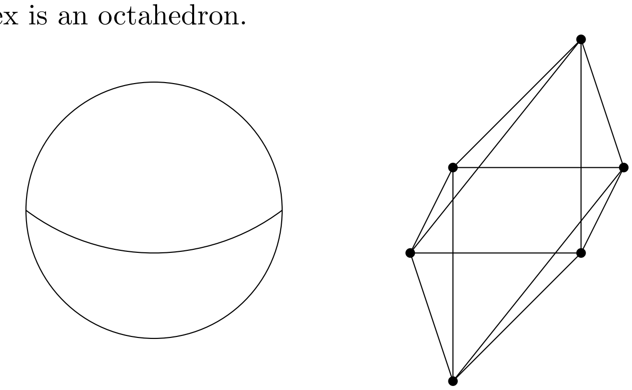
Theorem 3.6. *Every compact* 2*-dim manifold has triangulations.*

4. September 4, 2015

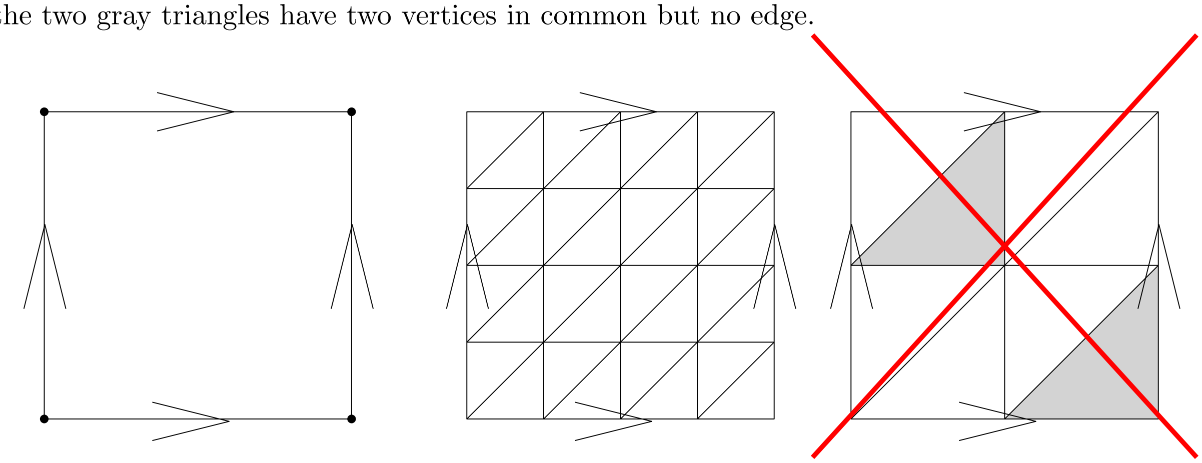
4.1. **Review.** Triangulations

Example 4.1. The 2-sphere is relatively easy to triangulate. Take three circumferences and their points of intersection.

The resulting complex is an octahedron.



Example 4.2. The torus is a bit harder to triangulate. The triangulation on the right fails since the two gray triangles have two vertices in common but no edge.



Definition 4.3. An Euler characteristic of a triangulation is given by $\chi(T) = V - E + F$

Theorem 4.4. *The Euler characteristic of a triangulation depends only on the homeomorphism class of the manifold.*

Proposition 4.5. $\chi(H(g)) = 2 - 2g$ *and* $\chi(M(g)) = 2 - g$.

Proof. The proof will only come much later. □

4.2. **(Geometric) Simplicial Complexes.**

Definition 4.6. Let $u_0, \dots, u_k \in \mathbb{R}^d$. An affine combination of $u_0, \dots, u_k$ is

$$x = \sum_{i=0}^{k} \lambda_i u_i; \qquad \lambda_i \in \mathbb{R}$$

with the condition $\sum_{i=0}^{k} \lambda_i = 1$.

The set of affine combinations of two points is a line. The set of affine combinations of 3 (linearly independent) points is a 2-plane.

Definition 4.7. The *affine hull* of $u_0, \dots, u_k$ is the set of all possible affine combinations.

Definition 4.8. The points $u_0, \dots, u_k$ are *affinely independent* if

$$\sum_{i=0}^{k} \lambda_i u_i = \sum_{i=0}^{k} \mu_i u_i \iff \underline{\lambda} = \underline{\mu} \in \mathbb{R}^{k+1}.$$

Remark 4.9. The points $u_0, \dots, u_k$ are affinity independent if and only if $v_i = u_i - u_0$ for $i = 1, \dots, k$ are linearly independent.

Corollary 4.10. *There are at most $(d+1)$ affinely independent points in $\mathbb{R}^d$.*

If $k \leq d+1$ then the set of points $\{u_0, \dots, u_k\} \subset \mathbb{R}^{d(k+1)}$ that are dependent has zero measure (in the standard measure on that space).

Definition 4.11. A *convex combination* of $u_0, \dots, u_k$ is a point $\sum_{i=0}^{k} \lambda_i u_i$, where $\sum_{i=0}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.

Definition 4.12. A convex hull of $u_0, \dots, u_k$ is

$$\operatorname{conv}\{u_0, \dots, u_k\} = \left\{ \sum_{i=0}^{k} \lambda_i u_i : \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

Example 4.13. The convex hull of two points is a line segment.

The convex hull of three points is a triangle.

This assumes the points are not affinely independent.

Definition 4.14. Assume $u_0, \dots, u_k \in \mathbb{R}^d$ are affinely independent.

$S = \operatorname{conv}\{u_0, \dots, u_k\}$ is called a simplex. Define the dimension of S to be k.

The empty simplex is a simplex by convention, with dimension -1.

Definition 4.15. A face of a simplex $S = \operatorname{conv}\{u_0, \dots, u_k\}$ is a simplex $T = \operatorname{conv}\{u_{\alpha_0}, \dots, u_{\alpha_k}\}$ where $\alpha \subseteq \{0, 1, \dots, k\}$.

Exercise 4.16. For all $x \in S$, x is in the interior of exactly one face of S.

For this we need to define the boundary $\operatorname{bd}(S) = \{\cup_i T_i | T_i = \operatorname{conv}\{U_j | j \neq i\}\}$ Then the interior of the face is $\operatorname{int}(S) = S \setminus \operatorname{bd}(S)$.

Proof. Let $x \in S$. This implies that there exist $\lambda_0, \dots, \lambda_k$ such that $x = \sum_{i=0}^{k} \lambda_i u_i$. Then $T =$ unique face of S such that $x \in \operatorname{int}(T)$ and $\alpha = \operatorname{supp}(\lambda) = \{i \ | \lambda_i > 0\}$. □

Definition 4.17. A *(geometric) simplicial complex* is a collection $K = \{S_a\}$ of simplices, such that

(1) If $T \leq S$, $S \in K \Rightarrow T \in K$.
(2) If $S_1, S_2 \in K$ then $S_1 \cap S_2$ is a face of both S_1 and S_2, where we consider the empty set to be a face of every simplex.

The dimension of K is defined as the maximal dimension of its faces. The underlying space $|K| = \bigcup_{S \in K} S =$ the underlying space with the induced topology.

Definition 4.18. The triangulation of a topological space X is a pair $(K, f : K \to X)$ where K is a geometric simplicial complex and $f : K \to X$ is a homeomorphism.

Sales pitch: When we have a triangulation, everything about the topology of X is encoded in the combinatorics of K.

Definition 4.19. An *abstract simplicial complex* ... will be defined next class.

5. September 9, 2015

Definition 5.1. Let V be a set, then a collection of subsets $A \subset 2^V$ will be called an abstract simplicial complex if it is closed downward, i.e. if $\sigma \in A$ and $\tau \subset \sigma$ then $\tau \in A$.

Example 5.2. The following are abstract simplicial complexes: $A = \varnothing$ – no subsets; $A = \{\varnothing\}$ – not empty: it contains the set $\varnothing$. $A = \{\varnothing, \{1\}$ with the ambient set $V = \{1\}$.

An example of a non-simplicial complex is $A = \{\{1\}, \{1, 2\}\}$ – this is not simplicial because even though $\{2\} \subset \{1, 2\}$, we do not have $\{2\} \in A$.

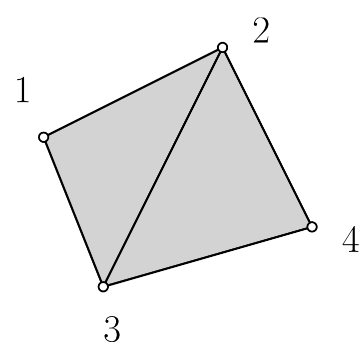
Remark 5.3. For any geometric simplicial complex there exists a unique abstract simplicial complex such that

$$K = \{S(\alpha) = \text{conv}\{p_i\}_{i\in\alpha}\}$$

V is defined as the set of 0-dimensional simplices.

Then $A = \{\alpha \in 2^V \mid \exists S \in K : S = \text{conv}\{p_i\}_{i\in\alpha}\}$.

Example 5.4. Consider the following geometric simplicial complex.



Here $V = \{1, 2, 3, 4\}$, $A \subset 2^V$ is given by $A =$ the subsets of $\{1, 2, 3\}$ and $\{2, 3, 4\}$.

Definition 5.5. Such an abstract simplicial complex is called the vertex scheme.

Remark 5.6. If $p_i \in \mathbb{R}^N$. Denote $S(\alpha) = \text{conv}\{p_i\}_{i\in\alpha}$.

Abstract	**Geometric**
$\beta \subseteq \alpha$	$T(\beta) \leq S(\alpha)$
V	vertices of K
$\dim S = \text{card}(\alpha) - 1$	$\dim S = d$
$\dim A = \max_{\alpha\in A}(\dim \alpha)$	$\dim A = \max_{S\in A} \dim S$

Table 1. Analogous Properties of Abstract and Geometric Simplicial Complexes

Theorem 5.7 (Geometric Realization Theorem)**.** *Let A be an abstract simplicial complex of $\dim A = d$ then there exists a geometric realization in $(2d+1)$-dimensional space.*

Remark 5.8. $2d+1$ is a tight condition for all d. There exist examples of complexes not realizable in dimension $2d$. For example with $d = 2$, the complete graph K_5 is a 1-dimensional complex; since it is nonplanar, it cannot be embedded in dimension $2d = 2$ without self-intersections. The rules of geometric simplicial complexes however demand that all intersections of faces are themselves faces of the complex.

Lemma 5.9. *Any $(m+1)$ distinct points*

$$\begin{array}{rl} & \gamma(t_0), \gamma(t_1), \ldots, \gamma(t_m) \\ \textit{where} & \gamma(t) = (t, t^2, \ldots, t^m) \end{array}$$

are affinely independent if and only if $t_i \neq t_j$.

Proof. The determinant given by:

$$\det \begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^m \\ 1 & t_1 & t_1^2 & \cdots & t_0^m \\ \vdots & & & \vdots & \\ 1 & t_m & t_m^2 & \cdots & t_m^m \end{pmatrix} = \prod_{0 \leq i < j \leq m} (t_j - t_i)$$

is the Vandermonde determinant which is only zero if two t-values are the same. □

Corollary 5.10. *For every finite set V, there exists a map $p : V \to \mathbb{R}^{2d+1}$ such that any $k \leq 2d+2$ are affinely independent.*

Proof. $A \subset 2^V$ is an abstract simplicial complex with $V =$ the set of vertices and $\dim A = d$ given by the maximal cardinality of a face of A.

For each $r \in V$, we have $p_r \in \mathbb{R}^{2d+1}$ such that any $2d+2$ points are affinely independent.

We can define $\forall \alpha \in A$:

$$S(\alpha) := \operatorname{conv}\{p_r\}_{r \in \alpha}.$$

This is always a simplex because the points are affinely independent.

Now we need to confirm the simplicial complex axioms.

(1) S is a simplex.
(2) $T \leq S, S \in K \implies T \in K$. (True because if $\alpha \in A, \beta \subset \alpha \implies \beta \in A$.)
(3) $S_1, S_2 \in K$, then $S_1 \cap S_2$ is either empty or a face of each.

The first two are trivial. Proving (2), let $S_1 = S(\alpha_1), S_2 = S(\alpha_2)$.

$$\begin{array}{rcl} \operatorname{card}(\alpha_1 \cup \alpha_2) & = & \operatorname{card}(\alpha_1) + \operatorname{card}(\alpha_2) - \operatorname{card}(\alpha_1 \cap \alpha_2) \\ \implies \operatorname{card}(\alpha_1 \cup \alpha_2) & \leq & (d_1 + 1) + (d_2 + 1) \\ & \leq & 2d + 2 \end{array}$$

Thus conclude that the vertices are affinely independent. We need to show: $X \in S_1 \cap S_2 \implies X$ is a face of S_i. Recall that a convex combination of affinely independent points has a unique formulation. Thus there is a specific $\beta_1 \subset \alpha_1$ and $\beta_2 \subset \alpha_2$, such that $X = \sum y_r p_r$, and $\beta_1 = \beta_2 = \operatorname{supp} y$; in particular $\beta_1 = \beta_2 = \alpha_1 \cap \alpha_2$. □

6. September 11, 2015

The geometric realization theorem sets up a correspondence between abstract simplicial complexes and geometric simplicial complexes.

Let K, L be two (geometric) simplicial complexes.

Definition 6.1 (1)**.** A PL-map $f : K \to L$ is a map defined on each simplex of K as:

$$f\left(\sum_{i=0}^{k} \alpha_i U_i\right) = \sum_{i=0}^{k} \alpha_i f(U_i)$$

PL stands for *piecewise linear.*

Note that the map is uniquely specified by the values on the vertices.

Definition 6.2 (1*)**.** Let $A \subset 2^V, B \subset 2^U$ be two abstract simplicial complexes.
A simplicial map is a map $m : A \to B$ that satisfies $\forall \sigma \in A$,

$$\sigma = (i_0, i_1, \ldots, i_k) = \bigcup_{j=0}^{k} \{i_j\} \qquad \Rightarrow m(\sigma) = (i_0, i_1, \ldots, i_k) = \bigcup_{j=0}^{k} m(\{i_j\}).$$

Definition 6.3 (1**)**.** Let A, B be a simplicial complex with $V = vert(A), U = vert(B)$, then a map $m_0 : V \to U$ is simplicial if $\forall \sigma \in A$,

$$\bigcup_{V \in \sigma} m_0(V) \in B$$

Remark 6.4. The following diagram commutes:

$$\begin{array}{ccc} K & \longrightarrow & \text{vertex scheme } A_K \\ \downarrow {\scriptstyle PL} & & {\scriptstyle m} \downarrow {\scriptstyle \text{simplicial map}} \\ L & \longrightarrow & \text{vertex scheme } A_L \end{array}$$

Definition 6.5 (2)**.** A PL map is a PL homeomorphism if it is a bijection on each simplex.

Definition 6.6 (2*)**.** A simplicial map is a simplicial complex isomorphism iff m_0 is a bijection.

Example 6.7. The image on the left and the right are not isomorphic as simplicial complexes but a subdivision of the left complex – given by the central complex is isomorphic to the one at right.

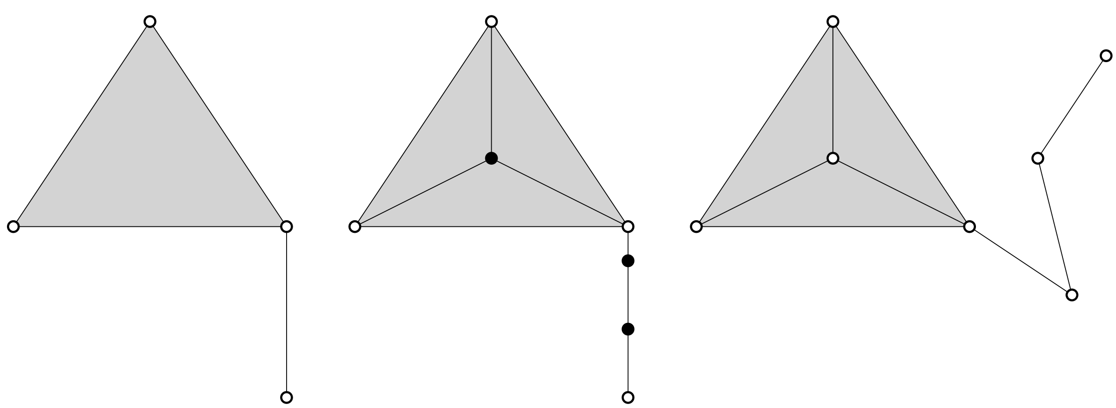


FIGURE 1. Subdivision of the Simplicial Complex Yields Isomorphism

Definition 6.8. A subdivision of a geometric complex adds in faces as in Example 6.7

Conjecture 6.9 (This was FALSE!)**.** Two compact manifolds are isomorphic if and only if their traingulations have isomorphic schemata after a finite number of subdivisions.

Theorem 6.10. *This conjecture holds for* $\dim M \leq 3$.

Definition 6.11. Let $A \subset 2^V$ be an abstract simplicial complex, then $Sd(A)$, the *barycentric subdivision*, is a simplicial complex $Sd(A) \subset 2^{A \setminus \varnothing}$ where $V \subset A$ is in $Sd(A) \iff V = \{\sigma_0, \dots, \sigma_k\}$ such that $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k$.

Example 6.12. We perform barycentric subdivision of the 1-simplex and 2-simplex.

In general, if $K = \{S_a\}$, for each $S = \{u_0, \dots, u_k\}$ a simplex, introduce a new vertex

$$U_S = \frac{1}{k+1} \sum_{i=0}^{k} U_i$$

and define simplices according to the same rule as in the abstract simplicial complex.

Exercise 6.13 (Homework Qs)**.**
(1) Why is a Δ-complex not a triangulation?
(2) Why is a triangulation not a Δ-complex?
(3) What is the role of the vertex ordering in the Δ- complex induced by a triangulation?

7. September 18, 2015

No class on September 14, notes from Sep 16 to be posted later.

7.1. Simplicial Homology of Δ-complexes.
Let G be an abelian group.

Definition 7.1. The chain group

$$\Delta_n(X; G) = \{ \sum_{\sigma \text{ dim } n} a_\sigma \sigma \}$$

Without specified group, take

$$\Delta_n(X) = \Delta_n(X; \mathbb{Z}).$$

The boundary homomorphism maps:

$$\partial_n : \Delta_n(X; G) \to \Delta_{n-1}(X; G)$$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v_i}, \dots, v_n]}$$

The homology of a complex is the direct sum of the graded homology groups:

$$H_*(X; G) = \bigoplus_{n=0}^{\infty} H_n(X; G).$$

Now we return to the example of $\mathbb{RP}^2$:

Example 7.2. $H_*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$. We use the Δ-complex in Figure 2 to compute the homology.

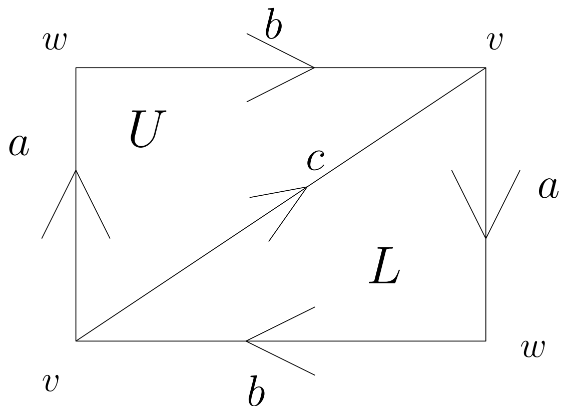


FIGURE 2. Δ-Complex for $\mathbb{RP}^2$

The chain groups at each step are given in this sequence:

$$0 \longleftarrow \Delta_0 \longleftarrow \Delta_1 \longleftarrow \Delta_2 \longleftarrow 0$$

$$0 \longleftarrow (\mathbb{Z}_2)^2 \longleftarrow (\mathbb{Z}_2)^3 \longleftarrow (\mathbb{Z}_2)^2 \longleftarrow 0$$

We can compute each kernel and image in order to find homology:

$$\begin{array}{rclrcl} \ker(\partial_2) & = & \langle U+L \rangle & \mathrm{Im}(\partial_3) & = & \langle 0 \rangle \\ \ker(\partial_1) & = & \langle a+b, c \rangle & \mathrm{Im}(\partial_2) & = & \langle a+b+c \rangle \\ \ker(\partial_0) & = & \langle v, w \rangle & \mathrm{Im}(\partial_1) & = & \langle w-v \rangle \end{array}$$

Therefore $H_*(\mathbb{RP}^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & * = 0,1,2 \\ 0 & \text{else.} \end{cases}$

Remark 7.3. If $A \subset 2^V$ is an abstract simplicial complex, then

$$C_n(A; G) = \{ \sum_{|\sigma|=n+1} a_\sigma \sigma | a_\sigma \in G \}$$

The boundary map and the homology groups are defined as before.

Moreover if $X = |A|$, the geometric realization of A, then $H_*(A, G) \cong H_*^\Delta(|A|, G)$; the abstract homology is the same as the Δ-complex homology.

7.2. **Singular Homology.**

Definition 7.4. A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \to X$.

Definition 7.5. Singular chains (with coefficients in G)

$$C_n(X; G) = \{ \sum_{\sigma \in I} a_\sigma \sigma | a_\sigma \in G \}$$

(only finitely many a_σ are nonzero; i.e. I finite).

The boundary homomorphism

$$\begin{array}{rccc} \partial_n : & C_n(X; G) & \to & C_{n-1}(X; G) \\ & \sigma & \mapsto & \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v_i}, \ldots, v_n]} \end{array}$$

Definition 7.6. $H^{Sing}(X;G) \cong \ker(\partial_n)/\operatorname{Im}(\partial_{n+1})$.

Theorem 7.7. $H_*^{\Delta}(X;G) \cong H_*^{Sing}(X;G)$.

Question 7.8. Why is this nicer to have?

The singular homology has nice functorial properties. For example, $f : X \to Y$ continuous induces $f_* : H_n(X) \to H_n(Y)$ group homomorphism.
For $a \in C_n(X;G)$ where $a = \sum_\sigma a_\sigma \sigma$; then $f_\sharp a = \sum_\sigma a_\sigma f_\sharp(\sigma)$.

Remark 7.9. The maps commute: $\partial_n f_\sharp(a) = f_\sharp \partial_n a$.

$$\begin{array}{ccc} C_n(X) & \xrightarrow{f_\sharp} & C_n(Y) \\ \big\downarrow \partial_n & & \big\downarrow \partial_n \\ C_{n-1}(X) & \xrightarrow{f_\sharp} & C_{n-1}(Y) \end{array}$$

Exercise 7.10 (Homework)**.** Prove that the map $f_\sharp : C_n(X) \to C_n(Y)$ is a group homomorphism that "extends" to $f_* : H_n(X) \to H_n(Y)$ via $f_*(a + \operatorname{Im}\partial_{n+1}) := f_\sharp a + \operatorname{Im}\partial_{n+1}$.

Proposition 7.11. *If $X \cong Y$ homeomorphic then if $f : X \to Y$ is a homeomorphism then $f_* : H_*(X) \to H_*(Y)$ is a group isomorphism.*

8. September 21, 2015

8.1. Last few classes.

- Simplicial homology $H_*^{\Delta}(X;G)$
- Singular homology $H_*^{sing}(X;G)$

The last theorem we discussed in class:

Proposition 8.1. *If $X \cong Y$ homeomorphic then if $f : X \to Y$ is a homeomorphism then $f_* : H_*(X) \to H_*(Y)$ is a group isomorphism.*

Definition 8.2. A graded abelian group is $C = \bigoplus_{i \in \mathbb{Z}} C_i$ where C_i are abelian groups.

Definition 8.3. A chain complex is a graded abelian group with group homomorphisms $\partial_i : C_i \to C_{i-1}$ such that $\partial_{i-1} \circ \partial_i = 0$.
$Z_i(C) = \ker(\partial_i : C_i \to C_{i-1})$ are cycles.
$B_i(C) = \operatorname{Im}(\partial_{i+1} : C_{i+1} \to C_i)$ are boundaries.
$H_i(C) = Z_i(C)/B_i(C)$.

Let $C_* = \bigoplus_i C_i$ and $D_* = \bigoplus_i D_i$ be chain complexes.

Definition 8.4. A chain map is a collection of group homomorphisms $f_i : C_i \to D_i$ such that the following diagram commutes:

$$\begin{array}{ccc} C_i & \xrightarrow{f_i} & D_i \\ \big\downarrow \partial_i & & \big\downarrow \partial_i \\ C_{i-1} & \xrightarrow{f_{i-1}} & D_{i-1} \end{array}$$

Lemma 8.5. *A chain map induces a group homomorphism $f_* : H_i(C) \to H_i(D)$.*

Proof. $H_i(C) = \ker\partial_i/\operatorname{Im}\partial_{i+1}$. Let $c \in C_i$ be a cycle such that $\partial c = 0$. Notation: $[c] := c + \operatorname{Im}\partial_{i+1} \in H_i(C)$.
Define $f_*([c]) = [f(c)] = f(c) + \operatorname{Im}\partial_{i+1} \in H_i(D)$. We need to show that:

(1) $\partial f(c) = 0$. [This follows from $\partial f = f\partial$.]
(2) If $\tilde{c} = c + \partial a$ then $[f(\tilde{c})] = [f(c)]$. [This follows by f being a group homomorphism.]

□

Corollary 8.6. *If $g : X \to Y$ is a continuous map of topological spaces, then $g_\sharp : C_i(X) \to C_i(Y)$ induces a group homomorphism*

$$g_* : H_i(X; G) \to H_i(Y; G)$$

Proof. $g_\sharp : C_i(X; G) \to C_i(Y; G)$ is a chain map. □

Lemma 8.7. *If $f : C_* \to D_*$ is a chain group isomorphism (i.e. $f_i : C_i \to D_i$ are group isomorphisms) and $\partial_i f_i = f_{i-1}\partial_i$, then*

$$f_* : H_i(C) \to H_i(D)$$

is a group isomorphism.

Proof. Chase some diagrams. □

Corollary 8.8. *If $g : X \to Y$ is a homeomorphism, then $g_* : H_i(X; G) \to H_i(Y; G)$ is a group isomorphism.*

Proof. $g_\sharp : C_i(X; G) \to C_i(Y; G)$ such that $\forall \sigma : \Delta^i \to X$
$g_\sharp(\sigma) = g \circ \sigma$ is a chain map. Notice that it has a chain map inverse.
Note that $(g_\sharp^{-1}) \circ (g_\sharp) = Id_{C_i(X;G)}$. Use the Lemma. □

Remark 8.9. The converse is not true. $H_*(S^1 \times \mathbb{R}^1) = H_*(S^1)$ but the spaces are not homeomorphic.

Another simple lemma:

Lemma 8.10. *Let C_* be a chain complex such that $C_i = \bigoplus_\alpha C_i^\alpha$ and $\partial_i C_i^\alpha \subseteq C_{i-1}^\alpha$.*
Then $H_i(C) = \bigoplus_\alpha H_i(C^\alpha)$.

Corollary 8.11. *If $X = \bigsqcup_\alpha X_\alpha$ where X_α are its path-connected components then*

$$H_i(X) = \bigoplus_\alpha H_i(X_\alpha).$$

Proof. Need to show that

$$C_i(X; G) =^? \bigoplus C_i(X_\alpha; G)$$

and $\partial C_i(X_\alpha, G) \subseteq C_{i-1}(X_\alpha, G)$.

$$C_i(X; G) = \{\sum_\sigma a_\sigma \sigma \mid a_\sigma \in G, \sigma : \Delta^i \to X\}$$

For each $\sigma : \Delta^i \to X$, observe that $\sigma(\Delta^i)$ must be path-connected thus lie in one of these X_α thus

$$C_i(X; G) \cong C_i(X_\alpha; G).$$

Note that if $\sigma : \Delta^i \to X_\alpha$ then $\partial\sigma \in C_{i-1}(X_\alpha; G)$. □

Definition 8.12. A chain complex is called an *exact sequence* if the homology is trivial.

Lemma 8.13. *If $0 \leftarrow A \leftarrow B \leftarrow C$ is an exact sequence then $A \cong B/C$.*

9. September 23, 2015

Corollary 9.1. *If X is path connected then $H_0(X;G) \cong G$.*

Proof. $H_0(X;G) \cong C_0(X;G)/\operatorname{Im}(\partial_1 : C_1(X;G) \to C_0(X;G))$.

Define ϵ such that

$$G \overset{\epsilon}{\leftarrow} C_0(X;G) \overset{\partial_1}{\leftarrow} C_1(X;G)$$

by sending $a \in C_0(X;G)$ where $a = \sum_\sigma a_\sigma \sigma$ where σ is a point, then $\epsilon(a) = \sum_\sigma a_\sigma$; i.e. add up all coefficients from the group. We claim that:

$$0 \leftarrow G \overset{\epsilon}{\leftarrow} C_0(X;G) \overset{\partial_1}{\leftarrow} C_1(X;G)$$

is an exact sequence. The proof of the corollary follows from this claim by Lemma 8.13.

$\boxed{\operatorname{Im}(\epsilon) = G.}$ Obvious.

$\boxed{\operatorname{Im}(\partial_1) \subseteq \ker(\epsilon).}$ Let $a = \partial_1(b)$ then $\epsilon(a) = \epsilon(\partial_1(b))$ where $b = \sum_\sigma b_\sigma \sigma$ and $\sigma \in C_1$ are one-dimensional simplices. $\epsilon(a) = \epsilon(\partial_1(b)) = \epsilon(\partial_1(\sum_\sigma b_\sigma \sigma)) = \sum_\sigma b_\sigma \epsilon(\partial_1(\sigma))$ where $\sigma : [p_0, p_1] \to X$ are one-dimensional simplices.

$\boxed{\ker(\epsilon) \subseteq \operatorname{Im}(\partial_1).}$ Assume $a \in C_0(X;G)$ and $\epsilon(a) = 0$. Want to Show: $\exists b$ such that $a = \partial_1 b$. Note that $a = \sum_\sigma a_\sigma \sigma$ where σ is a zero-dimensional simplex, i.e. a point.

Pick any x_0 then there exists a path from x_0 to each x_i corresponding to σ since X is path-connected. Indeed for each σ there exists $p_\sigma : [0,1] \to X$ with $p_\sigma(0) = x_0$ and $p_\sigma(1) = x_\sigma$. Define $b = \sum_\sigma a_\sigma p_\sigma$ Then

$$\begin{array}{rcccc} \partial_1(b) & = & \partial_1(\sum_\sigma a_\sigma p_\sigma) & = & \sum_\sigma a_\sigma \partial_1(p_\sigma) \\ & = & \sum_\sigma a_\sigma(p_\sigma|_1 - p_\sigma|_0) & = & \sum_\sigma a_\sigma x_\sigma - (\sum_\sigma a_\sigma) x_0 \\ & = & a - 0 \cdot x_0 & = & a \end{array}$$

Therefore $a \in \operatorname{Im} \partial_1$. □

10. September 25, 2015

10.1. **Last class.** We began the Mayer-Vietoris sequence. Short exact sequence $\Longrightarrow$ long exact sequence.

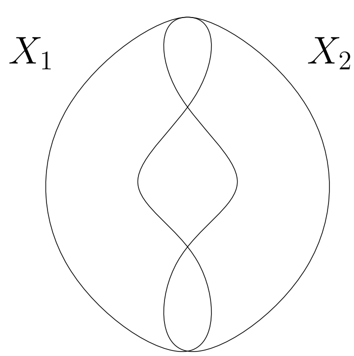


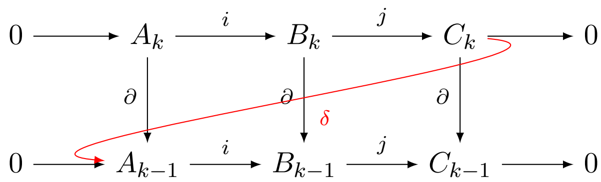
Figure 3. Union of topological spaces

Consider the union of spaces in Figure 3. It has a short exact sequence:

$$0 \to C_k(X_1 \cap X_2) \to C_k(X_1) \oplus C_k(X_2) \to C_k(X_1 \cup X_2) \to 0.$$

Question 10.1. If we understand $H_*(X_i)$ and $H_*(X_1 \cap X_2)$, what is $H_*(X_1 \cup X_2)$?

More generally, consider the commutative diagram of short exact sequences given below.



Specify some group elements as in this diagram:

$$\begin{array}{ccc} & & b & \xrightarrow{j} & c \\ & & \downarrow \partial & & \\ a & \xrightarrow{i} & \partial b & & \end{array}$$

Lemma 10.2. *There is a group homomorphism (connecting homomorphism) $\delta : H(C_k) \to H(C_{k-1})$ such that $\delta([c]) = [a]$, where a is defined as above.*

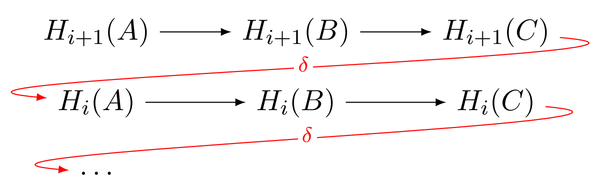
Proof. Need to show:

(1) $\partial a = 0$.
(2) Independent of choice of b.

For (1), we see that $i(\partial a) = \partial(ia) = \partial\partial b$ thus $\partial a = 0$ by injectivity of i.

For (2), assume a choice of $\tilde{b}$ such that $j\tilde{b} = c$. $\tilde{a} = i^{-1}(\partial\tilde{b})$ Wanted: $[\tilde{a} - a] = 0$. This means $\tilde{a} - a = \partial a'$. $i(\tilde{a} - a) = \partial\tilde{b} - \partial b = \partial(\tilde{b} - b)$. Simply set a' to be $i^{-1}(\tilde{b} - b)$ and the result has $\partial a' = \tilde{a} - a$ by injectivity. □

Theorem 10.3 (Short $\to$ long)**.** *Let $0 \to A_* \to B_* \to C_* \to 0$ be an exact sequence of chain complexes. Then, there is a long exact sequence:*



Proof. Remark:

$$i_*[a] = [ia] \qquad j_*[b] = [jb]$$

Need to prove:

$$\begin{array}{cc} \operatorname{Im} i_* \subseteq \ker j_* & \ker j_* \subseteq \operatorname{Im} i_* \\ \operatorname{Im} j_* \subseteq \ker \partial & \ker \partial \subseteq \operatorname{Im} j_* \\ \operatorname{Im} \partial \subseteq \ker i_* & \ker i_* \subseteq \operatorname{Im} \partial \end{array}$$

The left-hand containments prove that we have a chain complex, while the right-hand containments prove that it is exact. Diagram-chasing ensues. □

11. September 28, 2015

11.1. **Last class.** Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. Then there is a theorem:

Theorem 11.1. *There is a long exact sequence:*

$$H_{i+1}(A) \xrightarrow{i_*} H_{i+1}(B) \xrightarrow{j_*} H_{i+1}(C) \xrightarrow{\delta} H_i(A) \xrightarrow{i_*} H_i(B) \xrightarrow{j_*} H_i(C) \xrightarrow{\delta} \cdots$$

What are exact sequences good for?

Example 11.2. In the case of Mayer-Vietoris, $A_k = C_k^{sing}(X_1 \cap X_2)$, $B_k = C_k^{sing}(X_1) \oplus C_k^{sing}(X_2)$, and $C_k = C_k^{sing}(X_1 \cup X_2)$.

If $H_k(X_1) = H_k(X_2) = 0$ for $k > 0$, then $H_k(X_1) \oplus H_k(X_2) = 0 \oplus 0 = 0$ for $k > 0$. So the exact sequence, i.e. the Mayer-Vietoris sequence tells us that

$$0 \to H_k(X_1 \cup X_2) \xrightarrow{\delta} H_{k-1}(X_1 \cap X_2) \to 0$$

should be exact. In particular these groups are isomorphic.

Example 11.3. Again we refer to Figure 3 from earlier.

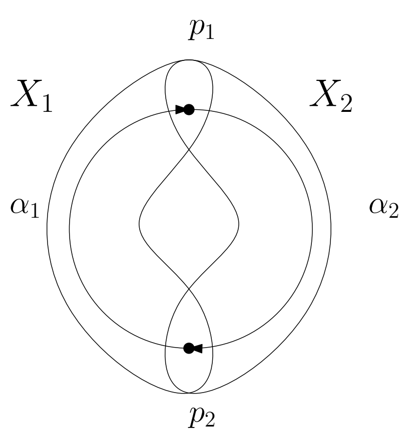


Figure 4. Chains in the Mayer-Vietoris Sequence

What does the map $\delta : H_1(X_1 \cup X_2) \to H_0(X_1 \cap X_2)$ do?

It maps a pair of 1-chains from $C_1^{sing}(X_1) \oplus C_1^{sing}(X_2)$ via $C_1^{sing}(X_1 \cup X_2) \to C_0^{sing}(X_1 \cup X_2) \to C_0^{sing}(X_1) \oplus C_0^{sing}(X_2)$ a pair of 0-chains. For a 1-chain to survive the homology functor it needs to have a single vertex. In other words the singular chain σ has $\sigma(0) = \sigma(1)$. Since $j(\alpha_1 \oplus \alpha_2) = \alpha_1 - \alpha_2$, we have $j(\alpha_1 \oplus \alpha_2) = \sigma$. This means $\partial\alpha_1 = [p_2] - [p_1] = \partial\alpha_2$. This means that $i^{-1}\delta([\alpha_1]) = [p_2] - [p_1]$. This specifies the value of $\delta([\sigma]) = [\beta]$.

Example 11.4 (Triangualtion of a sphere)**.** Let X_1 and X_2 be cones over the same triangle. Their intersection is a triangle. A sphere has nonzero H_2; here its generator would be $[\sigma]$ a signed sum of the six triangles. The map δ goes to the equator given by the intersection triangle.

11.2. **Homotopy equivalence. "I hid the truth from you."** Recall: $X \cong Y$ implies $H_*(X) = H_*(Y)$.

More generally:

Definition 11.5. Two continuous maps $f, g : X \to Y$ are called *homotopic* if there exists continuous functions $F : X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. The function $F : X \times [0,1] \to Y$ is called a *homotopy.*

Example 11.6. If f, g are functions from $\mathbb{R} \to \mathbb{R}$ then a homotopy is a 2-d surface in $\mathbb{R}^3$ as pictured in Figure 5.

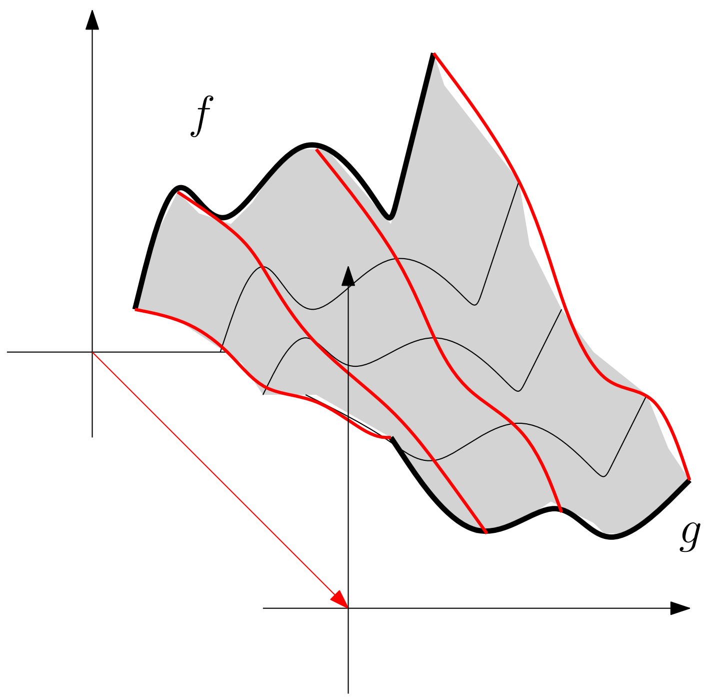


FIGURE 5. Homotopy from f to g.

Notation: $f \sim g$ means f is homotopic to g.

Lemma 11.7. *Homotopy is an equivalence relation on continuous maps. In particular,* $f \sim f, f \sim g \implies g \sim f$, *and* $f \sim g, g \sim h \implies f \sim h$.

Definition 11.8. $f : X \to X$ is *null-homotopic* if f is homotopic to id_X i.e. $f \sim id_X$.

Definition 11.9. Let $A \subseteq X$ be a subspace. A is called a *deformation retract* of X if it has a *deformation retraction*, a homotopy from id_X to a map sending $X \to A$ which is the identity on A.

Example 11.10. Take X to be the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$ in $\mathbb{R}^3$ and map $(x, y, z) \to (x, y, 0)$. The homotopy $F((x, y, z), t) = (x, y, (1-t)z)$ would be a deformation retraction.

Remark 11.11. If F is a deformation retraction, let $r(X) := F(x, 1)$. and $i : A \hookrightarrow X$ be the inclusion. Then $r \circ i = id_A$, and $i \circ r \sim id_X$.

Example 11.12. Any point is a deformation retract of $\mathbb{R}^n$.

Definition 11.13. X is *homotopy-equivalent* to Y $(X \sim Y)$ if $\exists f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.

Lemma 11.14. *Homotopy equivalence is an equivalence relation.*

Example 11.15. For all n, $\mathbb{R}^n \sim_{homotopy}$ a point. Why? If A is a deformation retract of X, then $A \sim_{homotopy} X$.

Theorem 11.16. *If $f, g : X \to Y$ are homotopic (i.e. $f \sim g$) then $f_* = g_*$ as maps of homology $H_*(X; G) \to H_*(Y; G)$.*

Corollary 11.17. *If $X \sim Y$, then $H_*(X) \cong H_*(Y)$.*

12. September 30, 2015

12.1. **Last class.** We saw what makes two maps $f, g : X \to Y$ homotopy-equivalent. We also defined homotopy-equivalent spaces to be connected by continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Theorem 12.1. *If $f, g : X \to Y$ are homotopic (i.e. $f \sim g$) then $f_* = g_*$ as maps of homology $H_*(X; G) \to H_*(Y; G)$.*

Corollary 12.2. *If $X \sim Y$, then $H_*(X) \cong H_*(Y)$.*

Proof.

$$\begin{array}{rll} & (f \circ g)_* = id_{H_k(Y;G)} & (g \circ f)_* = id_{H_k(X;G)} \\ \text{but} & (f \circ g)_* = f_* g_* = id_{H_k(Y;G)} & (g \circ f)_* = id_{H_k(Y;G)} \end{array}$$

Thus $f_* = g_*^{-1}$, which means we have group isomorphism. □

Remark 12.3. The converse of this Theorem is not true. In particular, there exist non-homotopy equivalent spaces with isomorphic homology groups.

Example 12.4 (3-sphere)**.** Y = the Poincare homology sphere. This has

$$H_n(Y; G) = \begin{cases} G & n = 0, 3 \\ 0 & n \notin \{0, 3\} \end{cases}$$

But Y has nontrivial fundamental group π_1. In fact $\pi_1(Y)$ is the icosahedral group.

Definition 12.5. Homotopy type is an element of the category topological spaces modulo the equivalence relation of being connected by a homotopy.

Definition 12.6. X is contractible if $X \sim$ point.
In particular, X is contractible implies $\widetilde{H}_*(X) = 0$.

Remark 12.7. The converse is not true.

The homology of X is determined by the homotopy type of X.
Let $A \subset 2^V$ be an abstract simplicial complex.

Definition 12.8. Homotopy type of $*$ is the homotopy type of its geometric realization.

Lemma 12.9. *The homotopy type of A does not depend on the choice of a geometric realization.*

Fact 12.10. *Even in the case of a finite abstract simplicial complex A i.e. $A \subset 2^V$ for $|V| < \infty$, there is no algorithm deciding contractibility.*
However if $\widetilde{H}_*(A) \neq \varnothing$, then A is not contractible.

12.2. **Nerves and Cech complexes.** Let $\mathcal{U} = \{U_v\}_{v \in V}$.

Definition 12.11. The nerve of $\mathcal{U}$ is an abstract simplicial complex nerve$(\mathcal{U}) \subset 2^V$ defined as nerve$(\mathcal{U}) = \{\sigma \subset V | \bigcap_{v \in \sigma} U_v \neq \varnothing\}$. Note that $\bigcap_{v \in \varnothing} U_v = X$.

Notation: $U_\sigma = \bigcap_{v \in \sigma} U_v$ is contractible.

Remark 12.12. This is an abstract simplicial complex i.e. $\nu \subset \sigma$, $\sigma \in$ nerve$(\mathcal{U}) \implies \nu \in$ nerve$(\mathcal{U})$.

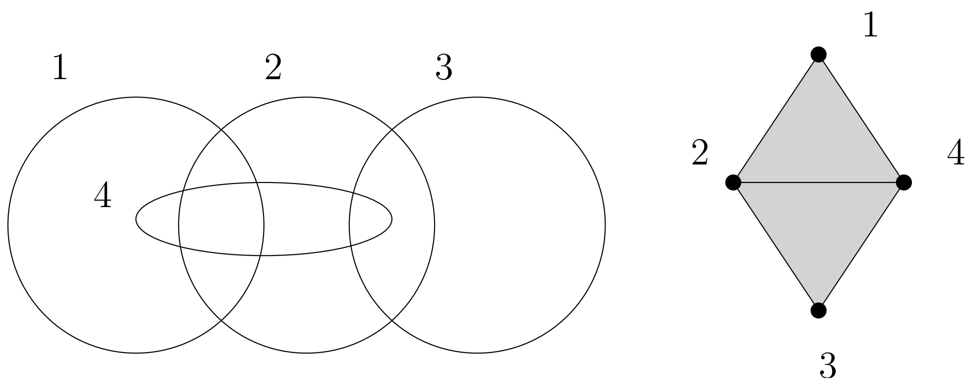


Figure 6. Nerve of a set arrangement

Definition 12.13. The collection of sets $\mathcal{U} = \{U_v\}_{v \in \mathcal{V}}$ is called a *locally finite cover* if:

(1) $\mathcal{U}$ is a cover, i.e. $\bigcup_{v \in V} U_v = X$.
(2) the cover is locally finite: $\forall x \in X$ there exists at most a finite number of U_v such that $x \in U_v$.

Theorem 12.14 (Nerve Lemma – Open Version)**.** *Assume that $\mathcal{U} = \{U_v\}_{v \in V}$ is a locally finite cover of a triangulable topological space X, and moreover:*

(1) $\mathcal{U}_v$ are open.
(2) U_σ is contractible for all $\sigma \in$ nerve$(\mathcal{U})$, for $\sigma \neq \varnothing$.

Then $X \sim_{homotopy}$ nerve$(\mathcal{U})$.

Theorem 12.15 (Nerve Lemma – Closed Version)**.** *Assume that $\mathcal{U} = \{U_v\}_{v \in V}$ is a finite cover of a triangulable topological space X, and moreover:*

(1) $\mathcal{U}_v$ are closed.
(2) U_σ is contractible for all $\sigma \in$ nerve$(\mathcal{U})$, for $\sigma \neq \varnothing$.

Then $X \sim_{homotopy}$ nerve$(\mathcal{U})$.

Example 12.16. All open or all closed cannot be relaxed. For instance, the interval can be split into an open interval and a closed interval, which means even though the interval is contractible, it has a cover with nerve two points.

Remark 12.17. In the closed case, the "finite" condition cannot be dropped either.

Example 12.18. Consider the unit circle $X = S^1$. Let

$$U_i = \{e^{2\pi i t} \mid \frac{1}{i+1} \leq t \leq \frac{1}{i}\}.$$

Claim: Homotopy type nerve$\{U_i\} \neq$ homotopy type of S^1.

Example 12.19. $S^1 \times [a, b] \sim_{hom} S^1$.

Remark 12.20. Contractibility of every intersection: If $X \subset \mathbb{R}^d$ is such that $U_i \subset X \subset \mathbb{R}^d$. If U_i are convex, then any intersection is also convex! Thus it is also contractible.

Convex $\implies$ contractible, since you can contract all points to a fixed point along lines.