1. Friday, August 24, 2012

1.1. **Squarefree Monomial Ideals.**

Definition 1.1. A *simplicial complex* Δ on $\{1, 2, \dots, n\}$ is a collection of subsets such that $\sigma \in \Delta$ and $\tau \subset \sigma \Rightarrow \tau \in \Delta$.

Example 1.2 (n = 5)**.** $\Delta =$ all subsets of $\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{5\}$. (See Figure 1.) The vector $f = (1, 5, 5, 1)$ indicates the number of sets in the simplicial complex with the given cardinality.

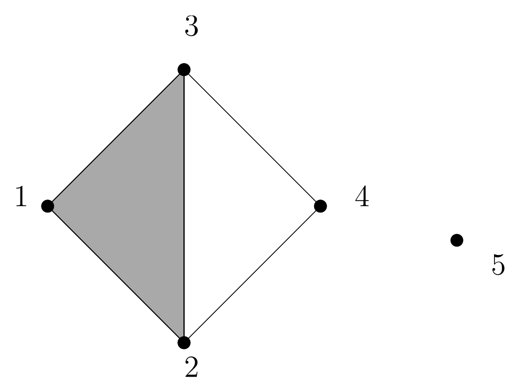

Figure 1. Simplicial Complex.

Definition 1.3. The *Stanley-Reisner ideal* of Δ is the monomial ideal

$$I_\Delta = \langle x^\tau : \tau \notin \Delta \rangle.$$

Remark 1.4. We identify subsets $\tau \subset \{1, 2, \dots, n\}$, vectors in $\{0, 1\}^n$ and squarefree monomials $x^\tau = \prod_{i \in \tau} x_i$.

Example 1.5. From the simplical complex Δ above, we have

$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_3x_4, x_2x_5, x_3x_5, x_4x_5 \rangle.$$

Theorem 1.6. *The map $\Delta \to I_\Delta$ is a bijection between simplicial complexes on $\{1, 2, \dots, n\}$ and square free monomial ideals in $S = K[x_1, \dots, x_n]$. Furthermore,*

$$I_\Delta = \bigcap_{\sigma \in \Delta} \langle x_i : i \notin \sigma \rangle.$$

The facets (minimal non-faces) suffice to generate the ideal.

Example 1.7. Again, from the above, we have

$$I_\Delta = \langle x_4, x_5 \rangle \cap \langle x_1, x_2, x_5 \rangle \cap \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_2, x_3, x_4 \rangle.$$

Definition 1.8. The *Alexander dual* Δ^* consists of the complements of the non-faces of Δ.

Example 1.9. We can construct I_{Δ^*} from the monomial generators of I_Δ or from the primary decomposition:

$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_3x_4, x_2x_5, x_3x_5, x_4x_5 \rangle$$
$$\Rightarrow I_{\Delta^*} = \langle x_1, x_4 \rangle \cap \langle x_1, x_5 \rangle \cap \langle x_2, x_3, x_4 \rangle \cap \langle x_2, x_5 \rangle \cap \langle x_3, x_5 \rangle \cap \langle x_4, x_5 \rangle.$$

$$I_\Delta = \langle x_4, x_5 \rangle \cap \langle x_1, x_2, x_5 \rangle \cap \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_2, x_3, x_4 \rangle$$
$$\Rightarrow I_{\Delta^*} = \langle x_4x_5, x_1x_2x_5, x_1x_3x_5, x_1x_2x_3x_4 \rangle.$$

Remark 1.10. In general, the choice of generators of an ideal is not unique; however, in the context of monomial ideals, a set of monomial generators will be unique.

1.2. **Hilbert Series ("Inclusion-Exclusion").**

Definition 1.11. An S- module M is $\mathbb{N}^n$*-graded* if $M = \bigoplus_{b \in \mathbb{N}^n} M_b$ and $x^a M_b \subseteq M_{a+b}$. Its *Hilbert Series* is:

$$H(M, \bar{x}) = \sum_{a \in \mathbb{N}^n} \dim_K(M_a) \cdot x^a.$$

Example 1.12.

$$H(S, x) = \prod_{i=1}^{n} \frac{1}{1 - x_i} = \text{ sum of all monomials in } S.$$

If I is a monomial ideal, then $H(S/I, X) =$ sum of all monomials not in I.

Definition 1.13. The K-polynomial of M is the numerator of

$$H(M, x) = \frac{K(M, x)}{\prod_{i=1}^{n}(1 - x_i)}.$$

Theorem 1.14. *The Stanley-Reisner ring has:*

$$K(S/I_\Delta, x) = \sum_{\sigma \in \Delta} \left(\prod_{i \in \sigma} x_i \prod_{j \notin \sigma} (1 - x_j) \right).$$

Example 1.15. For the square graph $abcd$:

$$I_\Delta = \langle ac, bd \rangle.$$

$$K = 1 - ac - bd + abcd.$$

We can use the theorem to calculate:

$$K = (1-a)(1-b)(1-c)(1-d) + a(1-b)(1-c)(1-d) + \cdots + a(1-b)(1-c)d.$$

Remark 1.16. The K-polynomial allows us to count the monomials by inclusion-exclusion. First we take out the minimal monomials, but then we have to put back in the intersection, etc.

Corollary 1.17 (Stanley's Green Book)**.** *If* $d = \dim(\Delta) + 1$ *then*

$$H(S/I_\Delta; t, t, \ldots, t) = \frac{1}{(1-t)^n} \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{n-i}$$

$$= \frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

1.3. **Simplicial Complexes and Homology.**

Definition 1.18. For Δ a simplical complex on $[n] = \{1, 2, \ldots, n\}$, $F_i(\Delta = \{i$-dimensional faces of $\Delta\}$. The reduced chain complex of Δ over K is the sequence of linear maps:

$$0 \leftarrow K^{F_{-1}(\Delta)} \overset{\partial_1}{\leftarrow} K^{F_0(\Delta)} \overset{\partial_2}{\leftarrow} \cdots \overset{\partial_{i-1}}{\leftarrow} K^{F_{i-1}(\Delta)} \overset{\partial_i}{\leftarrow} K^{F_i(\Delta)} \overset{\partial_{i+1}}{\leftarrow} \cdots K^{F_{n-1}(\Delta)} \leftarrow 0.$$

where $\partial_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j, \sigma) e_{\sigma \setminus j}$, (Note: $\partial_{i-1} \circ \partial_i = 0$) and $\operatorname{sign}(j, \sigma) = (-1)^{r-1}$ if j is the r-th element in the sorted order of σ.

Definition 1.19. The i-th reduced homology of Δ over K is the vector space

$$H_i(\Delta) = \ker(\partial_i) / \operatorname{Im}(\partial_{i+1}).$$

Example 1.20 (Triangle Graph).

$$0 \leftarrow K^1 \xleftarrow{[1\,1\,1]} K^3 \xleftarrow{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}} K^3 \leftarrow 0.$$

Definition 1.21. The i-th reduced cohomology of Δ over K is the vector space

$$H^i(\Delta) = \ker(\partial^{i+1}) / \operatorname{Im}(\partial^i).$$

2. Monday, August 27, 2012

2.1. **Preview of Frobenius Splitting.** Let R be a commutative ring containing a perfect (such that every finite extension is separable) field K of characteristic p.

Freshmen's Dream describes a field where $(a+b)^p = a^p + b^p$. The Frobenius map $F : R \to R$, $a \mapsto a^p$ is K-algebra homomorphism.

Say R is Frobenius-split if the inclusion $FR \subset R$ of R-modules has a splitting, i.e. $\exists K$-linear map $\varphi : R \to R$ such that $\varphi(a^p \cdot b) = a\varphi(b)$. In other words, it can extract p-th roots.

Theorem 2.1 (Schwede, Knutson). *If this holds, then there are only finitely many ideas $I \subset R$ that are compatibly split, i.e. $\varphi(I) \subseteq I \Leftrightarrow \varphi(I) = I$.*

2.2. **Monomial Matrices.** Consider a sequence of $\mathbb{N}^n$-graded S-modules

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_{e-1}} F_{e-1} \xleftarrow{\phi_e} F_e \leftarrow 0.$$

Each ϕ_i preserves $\mathbb{N}^n$ grading: it can be written as a matrix with entries in K and row/column labels in $\mathbb{N}^n$.

Example 2.2. The sequence

$$0 \leftarrow S \xleftarrow{[\, ac \quad bd \,]} S^2 \xleftarrow{\begin{bmatrix} bd \\ -ac \end{bmatrix}} S \leftarrow 0.$$

is represented as follows in terms of monomial matrices:

$$0 \leftarrow S_{0000} \xleftarrow{[\, 1 \quad 1 \,]} S^2_{\substack{1010 \\ 0101}} \xleftarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} S_{1111} \leftarrow 0.$$

Definition 2.3.
- $\mathcal{F}_\bullet$ is a *complex* if $\phi_i \circ \phi_{i+1} = 0 \forall i$.
- $\mathcal{F}_\bullet$ is *exact* in homological degree i if $\ker(\phi_i) = \operatorname{im}(\phi_{i+1})$.
- $\mathcal{F}_\bullet$ is a *free resolution* of M if it is exact everywhere except in homological degree 0, where $M = F_0/\operatorname{im}(\phi_1)$.

Theorem 2.4 (Hilbert's Syzygy Theorem). *There exists such a free resolution of length $\leq n$ (the length of the grading).*

Remark 2.5. We can use this theorem to compute K-polynomials.

Example 2.6.

$$M = \frac{K[x_1, \ldots, x_4]}{\langle x_1x_3, x_2x_4 \rangle}$$

has $K = 1 - x_1x_3 - x_2x_4 + x_1x_2x_3x_4$.

Recommended Exercise for Homological Algebra: Exercise 1.12 – Find a direct proof that $Tor_i^S(M,N) \cong Tor_i^S(N,M)$.

2.3. **Betti Numbers.**

Definition 2.7. If $\mathcal{F}_\bullet$ is a minimal free resolution of M and $F_i = \bigoplus_{a\in\mathbb{N}^n}(S_a)^{\beta_{i,a}}$, then the *i-th Betti number of M* in degree a is $\beta_{i,a} = \beta_{i,a}(M)$.

Remark 2.8.

$$K(M,x) = \sum_{a\in\mathbb{N}}\sum_{i=0}^{l}(-1)^i\beta_{i,a}(M)x^a.$$

Question 2.9. Can we recover the resolution from knowing $K(M,x)$?

Answer. Usually no, but sometimes yes. ☐

Lemma 2.10.

$$\beta_{i,a}(M) = \dim_K Tor_i^S(K,M)_a.$$

Here K is the $S-$module described by $S/\langle x_1,\ldots,x_n\rangle$.

Definition 2.11. For a (not necessarily squarefree) monomial ideal I and degree $b\in\mathbb{N}^n$, define the *Koszul simplicial complex*

$$\mathcal{K}^b(I) = \{\tau \,|\, x^{b-\tau}\in I\}.$$

Theorem 2.12 (Hochster)**.** *The Betti numbers of I and S/I in degree b can be expressed as*

$$\beta_{i,b}(I) = \beta_{i+1,b}(S/I) = \dim_K \widetilde{H}_{i-1}(\mathcal{K}^b(I);K).$$

(Here, we are discussing reduced homology.)

Ideas of Proof. $\beta_{i,b}(I)$ is the i-th homology of $\mathbb{K}_\bullet\otimes I$ in degree b. Since I is a submodule of S, the complex $(\mathbb{K}_\bullet\otimes I)_b$ is a subcomplex of $(\mathbb{K}_\bullet)b$ = chain complex of the simplex on $\sigma = supp(b)$. It consists of all $\tau\subseteq\sigma$ such that $I(-\tau)_b\neq 0 \Leftrightarrow I$ is nonzero in degree $b-\tau \Leftrightarrow x^{b-\tau}\in I$. ☐

Corollary 2.13. *The link of σ inside the simplicial complex Δ is*

$$link_\Delta(\sigma) = \{\tau\in\Delta | \tau\cup\sigma\in\Delta \text{ and } \tau\cap\sigma=\emptyset\}.$$

Example 2.14. The link of a vertex in ∂ Octahedron is a 4-cycle.

Corollary 2.15. *All nonzero Betti numbers of I_Δ and S/I_Δ lie in squarefree degrees σ where*

$$\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(S/I_\Delta) = \dim_K \widetilde{H}_{i-1}(link_{\Delta^*}(\bar\sigma);K).$$

Specifically, $\mathcal{K}^1(I_\Delta)=\Delta^*$, and its homology is given by the Betti numbers of I_Δ in degree 1.

Exercise 2.16. Use this to calculate the Alexander dual. See Figure 2.

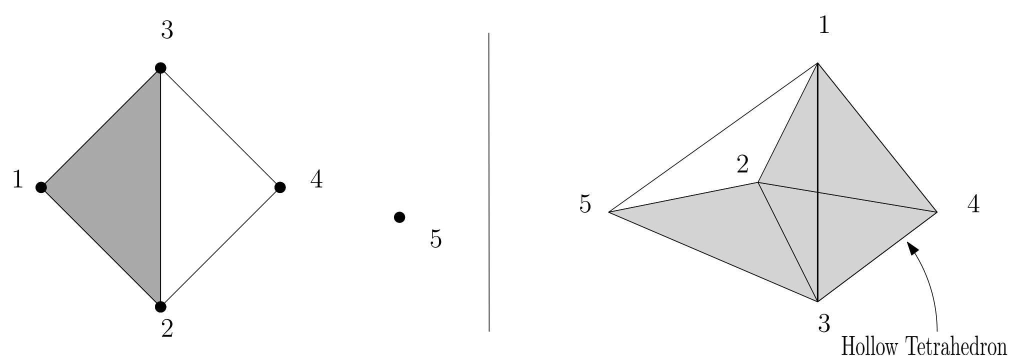

FIGURE 2. Simplicial Complex and its Alexander Dual.

$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_3x_4, x_2x_5, x_3x_5, x_4x_5\rangle$$
$$I_{\Delta^*} = \langle x_4x_5, x_1x_2x_5, x_1x_3x_5, x_1x_2x_3x_4\rangle.$$

See the Macaulay2 file for Betti diagrams. The 1's on the outer corners represent the interesting homology.

3. Wednesday, August 29, 2012

Let $S = K[x_1, \dots, x_n]$ be a ring with $\mathbb{N}$ grading, such that $\mathrm{char}(K) = 0$.

Let $GL_n(K) = \{$invertible $n\times n$ matrices$\}$, called the general linear group. Let $B_n(K) = \{$upper-triangular $n \times n$ matrices$\}$, called the Borel group. Let $T_n(K) = \{$diagonal $n \times n$ matrices$\}$, called the Torus group.

$$T_n(K) \subset B_n(K) \subset GL_n(K).$$

Proposition 3.1. *An ideal $I \subset S$ is fixed under T_n iff I is a monomial ideal.*

Proof by Example. Consider $f = 11x^2y + 17yz + 19xz^3 \in I \subset K[x, y, z]$, a torus-fixed ideal. Scale x, y, z by $2, 3, \dots$ (for example).

$$\begin{pmatrix} 11 & & \\ & 17 & \\ & & 19 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2^3 & 2^2 & 2^4 \\ 3^3 & 3^2 & 3^4 \end{pmatrix} \begin{pmatrix} x^2y \\ yz \\ xz^3 \end{pmatrix} \in \begin{pmatrix} I \\ I \\ I \end{pmatrix}$$

By fudging with the scaling numbers, you can get each entry to stand on its own. □

Proposition 3.2. *I is GL_n-fixed iff $I = \langle x_1, \dots, x_n\rangle^d$ for some $d \in \mathbb{N}$.*

Proposition 3.3. *For a monomial ideal I, the following are equivalent:*
1. *I is Borel-fixed.*
2. *If $m \in I$ is divisible by x_j, then $m\frac{x_i}{x_j} \in I$, for $i < j$.*

Fix a term order $<$ on S. If I is any ideal in S, then its generic initial ideal is

$$gin_<(I) := in_<(g \circ I).$$

where g is a random matrix in $GL_n(K)$, i.e. in a suitable Zariski open subset.

Theorem 3.4 (Theorem 15.20 in Eisenbud)**.** *$gin_<(I)$ is Borel-fixed.*

3.1. Gröbner Basis Review.

Example 3.5. Consider $I = \langle \Lambda^+_{(3)}\rangle$, under the lex order $x > y > z$. By definition,

$$I = \langle x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3\rangle.$$

The Gröbner basis is:

$$GB = \{x + y + z, y^2 + yz + z^2, z^3\}.$$
$$in_<(I) = \langle x, y^2, z^3\rangle.$$
$$S/I \cong_K K\{1, y, z, yz, z^2, yz^2\}.$$

Hilbert function ($\mathbb{N}$-grading):

d	0	1	2	3	4	5	6
S	1	3	6	10	15	21	28
S/I	1	2	2	1	0	0	0
I	0	1	4	9	15	21	28

$$gin_<(I) = \langle x, y^2, yz^2, z^4\rangle.$$

Example 3.6 (**GB for Submodules**)**.**

$$M = \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x+y \\ x+y \end{bmatrix} \right\rangle \subset S^2$$

TOP (term over position) Gröbner basis rules ties in favor of the term with greater weight. In our case,

$$GB = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix} \right\} \qquad in_<(M) = \langle xe_1, xe_2 \rangle.$$

$$GB = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ x^2-y^2 \end{bmatrix} \right\} \qquad in_<(M) = \langle xe_1, ye_1, (x^2-y^2)e_2 \rangle.$$

3.2. **Eliahou-Kervaire Resolution.**

Lemma 3.7. *Each monomial m in Borel-fixed monomial ideal*

$$I = \langle m_1, m_2, \ldots, m_r \rangle.$$

can be written uniquely as a product $m = m_i \cdot m'$, with $\max(m_i) \leq \min(m')$. Let $u_i = \max(m_i)$.

Proposition 3.8. *The $\mathcal{K}$-polynomial of S/I equals*

$$\mathcal{K}(S/I, x) = 1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{u_i-1} (1-x_j).$$

Example 3.9.

$$I = \langle x_1^2, x_1x_2, x_2^3, x_1x_3^3 \rangle.$$

$$\Rightarrow \mathcal{K} = 1 - x_1^2 - x_1x_2(1-x_1) - x_2^3(1-x_1) - x_1x_3^3(1-x_1) - x_1x_3^3(1-x_1)(1-x_2)$$

$$= 1 - x_1^2 - x_1x_2 - x_2^3 - x_1x_3^3 + x_1^2x_3^2 + x_1x_2x_3^3 + x_1x_2^3 + x_1^2x_2 - x_1^2x_2x_3^3.$$

This suggests the minimal free resolution

$$0 \longleftarrow S \longleftarrow S^4 \longleftarrow S^4 \longleftarrow S \longleftarrow 0.$$

Theorem 3.10. *Let $M \subset S^r$ be the module of first syzygies on a Borel-fixed monomial ideal I. Then M has a POT Gröbner basis whose initial module $in(M)$ has a linear free resolution. Moreover, $S^r/in(M)$ and $I \cong S^r/M$ have the same Betti numbers, namely:*

$$\beta_i = \sum_{j=1}^{r} \binom{u_j - 1}{i}.$$

Example 3.11 (Example 2.19 from Sturmfels-Miller)**.**

$$I = \langle x_1x_2x_4^4, x_1x_2x_3x_4^2, x_1x_3^6, x_1x_2x_3^2, x_2^6, x_1x_2^2, x_1^2 \rangle.$$

See text or Macaulay2 Code for more detail.

3.3. **Lex-Segment Ideals.** Fix a Hilbert function $H : \mathbb{N} \to \mathbb{N}$ of some homogeneous ideal $I \subset S$.
Let L_d be the K-span of the $H(d)$ largest monomials in the lex order on S_d and define

$$L = \bigoplus_{d=0}^{\infty} L_d.$$

Proposition 3.12 (Macaulay 1927)**.** *The graded vector space L is a Borel-fixed ideal.*

Remark 3.13. The part of this proposition that is non-trivial is to show that this vector space is an ideal of the ring.

Theorem 3.14 (Macaulay's Theorem)**.** *For every $d \in \mathbb{N}$, the lex-segment ideal L has at least as many generators as every other (monomial) ideal with the same Hilbert function H.*

Example 3.15. Intersect of a quadric and cubic surface in $\mathbb{P}^3$. As an ideal, this is generated by a quadratic and cubic homogeneous polynomial in 4 variables.

The Hilbert Series is:

$$\frac{1 - t^2 - t^3 + t^5}{(1-t)^4} = \frac{1 + 2t + 2t^2 + t^3}{(1-t)^2} = 1 + 4t + \sum_{r=2}^{\infty}(6r-3)t^r.$$

$$gin_{revlex}(I) = \langle x_1^2, x_1x_2^2, x_2^4\rangle \qquad gin_{lex}(I) = \langle x_1x_3^6, x_2^6, x_1x_2x_4, x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2\rangle.$$

The lex-segment ideal L has 18 generators, more than from any other term ordering.

Open problem: given two random homogeneous forms, compute generic initial ideal in the lex order.

Theorem 3.16 (Bigatti-Hulett Theorem)**.** *For all i and d, the lex-segment ideal L has the most degree d minimal i-th syzygies among all (monomial) among all (monomial) ideals with the Hilbert function H.*

4. Friday, August 31, 2012

4.1. **Staircases.** First, let us look at a 2-dimensional case.

Example 4.1. Consider the ideal $I \subset k[x, y]$ as follows.

$$I = \langle x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\rangle.$$

such that $a_1 > a_2 > \cdots > a_r$, and $b_1 < b_2 < \cdots < b_r$.

This can be portrayed in a staircase diagram as in Figure 3.

Let us consider the $\mathcal{K}$ polynomial of this ideal:

$$\mathcal{K}(S/I, x, y) = (1-x)(1-y)\sum\{x^iy^j \notin I\} = 1 - \sum_{i=1}^{r} x^{a_i}y^{b_i} + \sum_{j=1}^{r-1} x^{a_j}y^{b_{j+1}}.$$

The first sum corresponds to the inner corners, and the second sum to the outer corners.

Proposition 4.2. *The minimal free resolution of S/I equals*

$$0 \longleftarrow S \longleftarrow S^r \longleftarrow S^{r-1} \longleftarrow 0.$$

The minimal first syzygies are $(y^{b_{i+1}-b_i}e_i - x^{a_i - a_{i+1}}e_{i+1})$.

Proposition 4.3 (Irreducible Decomposition)**.**

$$I = \langle y^{b_1}\rangle \cap \langle x^{a_1}, y^{b_2}\rangle \cap \langle x^{a_2}, y^{b_3}\rangle \cap \cdots \cap \langle x^{a_r-1}, y^{b_r}\rangle \cap \langle x^{a_r}\rangle,$$

where the first (resp. last) intersectand is deleted if $b_1 = 0$ or $a_1 = 0$.

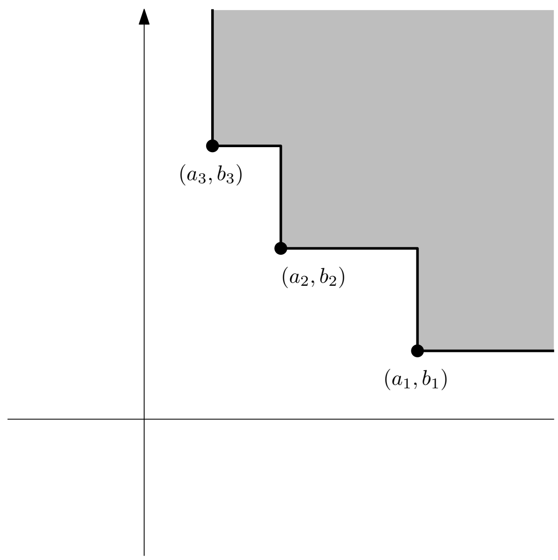

FIGURE 3. Staircase Diagram.

Definition 4.4. The *Buchberger graph* $Buch(I)$ of a monomial ideal $I = \langle m_1, m_2, \ldots, m_r\rangle$ has:

- Vertices $1, 2, \ldots, r$,
- Edges $\{i, j\}$ for i, j such that there is no k for which $m_k | lcm(m_i, m_j)$ and m_k has smaller degree in every variable occurring in $\operatorname{lcm}(m_i, m_j)$.

Proposition 4.5. *The module of syzygies on I is generated by the syzygies*

$$\sigma_{ij} = \frac{lcm(m_i, m_j)}{m_i} e_i - \frac{lcm(m_i, m_j)}{m_j} e_j$$

corresponding to edges in the Buchberger graph.

See Figure 3.2 in Sturmfels-Miller for an example. The ideal's decomposition can be read by breaking the cubical complex into cuboids. In labeling the Buchberger graph, in each face, you write the vector of the least common multiple of the vertices. The $\mathcal{K}$ polynomial is $1-$ (vertex labels) $+$ (edge labels) $-$ (face labels).

In general, the Buchberger graph is not planar. But it has nice properties under genericity conditions.

4.2. **Genericity and Deformations.**

Definition 4.6. A monomial ideal $I \subset K[X, Y, Z]$ is *strongly generic* if any generators $x^i y^j z^k$ and $x^{i'} y^{j'} z^{k'}$ have the property that $i \neq i'$ unless they are both zero, and similarly for the other indices.

Proposition 4.7. *If I is strongly generic, then the Buchberger graph is planar and connected. If I is also Artinian, then $Buch(I)$ consists of the edges of a triangulated triangle (thus, 3-connected).*

Definition 4.8. A planar map is a graph together with an embedding into a surface homeomorphic to $\mathbb{R}^2$.

Theorem 4.9. *Given a strongly generic monomial ideal I in $K[X, Y, Z]$, the planar map $Buch(I)$ provides a minimal free resolution of I:*

$$0 \longleftarrow S \longleftarrow S^r \xleftarrow{\partial_E} S^e \xleftarrow{\partial_F} S^f \longleftarrow 0.$$

The differentials in this sequence are:

$$\partial_E(e_{ij}) = \frac{m_{ij}}{m_j}e_i - \frac{m_{ij}}{m_i}e_j.$$

$$\partial_F(e_R) = \sum_{\text{edges } \{i,j\}\subset R} \pm\frac{m_R}{m_{ij}}e_{ij}, \text{ where } m_R = lcm(m_i|i\in R).$$

Question 4.10. What if I is not strongly generic?

Introduce a polynomial ring $S_\varepsilon = K[x^\varepsilon, y^\varepsilon, z^\varepsilon]$ for $\varepsilon = \frac{1}{N}$ for some $N \in \mathbb{N}$ which contains $S = K[X,Y,Z]$.

Consider monomial ideals:

$$I = \langle m_1, m_2, \ldots, m_r\rangle \subset S, \qquad \text{and } I = \langle m_{\varepsilon,1}, m_{\varepsilon,2}, \ldots, m_{\varepsilon,r}\rangle \subset S_\varepsilon.$$

Say I_ε is a strong deformation of I if the partial order on $\{1,2,\ldots,r\}$ by x-degree of the m_ε refines the partial order of the m_i and same for y and z.

Example 4.11.

$$I = \langle X,Y,Z\rangle^3.$$

We could approach this problem using Borel-fixed ideal theory, Eliahou-Kervaire resolutions, or even tropical strategy. We will use the Buchberger graph of I and I_ε. (See Figure Δ).

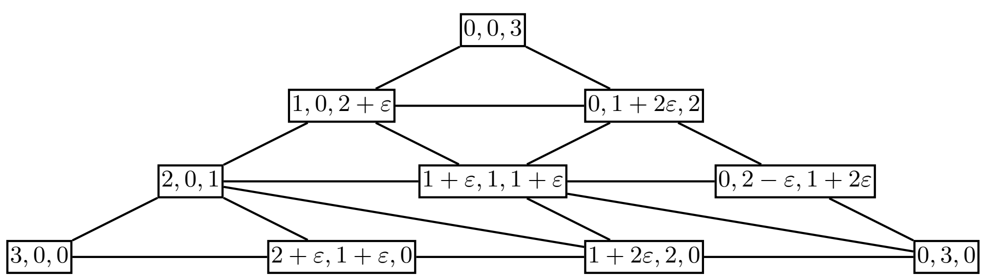

Figure 4. *Buch*(I_ε) after deformation.

Proposition 4.12. *Specializing the labels of the vertices, edges, and faces of the planar Buchberger graph Buch*(I_ε) *under* $\varepsilon = 0$ *yields a planar map resolution of* I*. (usually not minimal).*

Question 4.13. Can you always make it minimal?

Yes. See Section 3.5 of Sturmfels-Miller.

Corollary 4.14. *Let* r *be the number of generators of an ideal,* e *the number of first syzygies, and* f *the number of second syzygies. Then,* $e \le 3r-6$ *and* $f \le 2r-5$.

5. Wednesday, September 5, 2012

Definition 5.1. A *polyhedral complex* in $\mathbb{R}^m$ is a finite set X of convex polytopes such that

- If $P \in X$ and $F \subset P$ is a face, then $F \in X$.
- If $P, Q \in X$, then $P \cap Q$ is a face of both P and Q.

X has a (reduced) chain complex (over $\mathbb{Z}$) with boundary maps

$$\partial(F) = \sum_{\substack{\text{facets}\\ G\subset F}} \text{sign}(G,F)\cdot G.$$

where $\text{sign}(G,F) = 1$, if F's orientation induces G's orientation, or -1 otherwise.

Definition 5.2. X is a *labelled cell complex* if its r vertices are labelled by vectors $a_1, \ldots, a_r \in \mathbb{N}^n$. The label of any face $F \in X$ is given by

$$x^{a_F} = \operatorname{lcm}(x^{a_i} | i \in F).$$

The monomial matrix on X uses this chain complex for scalar entries with row and column labels a_F, for $F \in X$.

The *cellular free complex* $\mathcal{F}_X$ is the resulting complex of $\mathbb{N}^n$ graded free S-modules.

$$\mathcal{F}_X = \bigoplus_{\mathcal{F} \in X} S(-a_F). \qquad \partial(F) = \sum_{\substack{\text{facets} \\ G \subset F}} \operatorname{sign}(G, F) \cdot x^{a_F - a_G} \cdot G.$$

We call $\mathcal{F}_X$ a *cellular resolution* if it is exact.

Example 5.3. Consider the octahedron cell complex as in Figure 5. By counting the faces of various dimension, we obtain the cellular free complex below:

$$0 \leftarrow S \leftarrow S^6 \leftarrow S^{12} \leftarrow S^8 \leftarrow S \leftarrow 0.$$

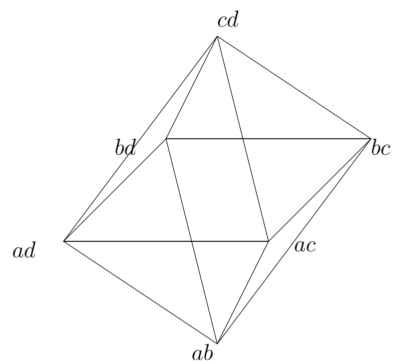

Figure 5. Labelled Cell Complex.

If Q is an order ideal in $\mathbb{N}^n$, then $X_Q = \{F \in X | a_F \in Q\}$ is a labeled sub complex of X.

Example 5.4. $X_{\preceq \mathbf{b}}$ and $X_{\prec \mathbf{b}}$.

Proposition 5.5. *$\mathcal{F}_X$ is a cellular resolution iff the cell complex $X_{\preceq \mathbf{b}}$ is acyclic over K for all $\mathbf{b} \in \mathbb{N}^n$.*

In this case, $\mathcal{F}_X$ resolves S/I where $I = \langle x^{a_v} | v \text{ vertex} \in F \rangle$.

Example 5.6 (Example 5.3 Continued)**.** Take $X_{\preceq abc}$, where X is the labeled cell complex from Example 5.3. The resulting subcomplex as depicted in Figure 6 is acyclic, as are all other subcomplexes. Therefore, $\mathcal{F}_X$ is a cellular resolution. However, it is not minimal, since the edge labels match the face label.

Theorem 5.7. *Write $\widetilde{H}_i(X, k)$ for the reduced homology. If $\mathcal{F}_X$ resolves I then*

$$\beta_{i,b}(I) = \dim_k \widetilde{H}_{i-1}(X_{\prec b}, k).$$

Example 5.8 (Example 5.6 Continued)**.** To calculate $\beta_{2,abcd}(I)$, we look at the subcomplex $X_{\prec abcd}$, depicted in Figure 7. The $\mathcal{K}$-polynomial of this ideal is:

$$\mathcal{K} = 1 - ab - ac - ad - bc - bd - cd + abc + abd + acd + bcd - 3abcd.$$

Because $\dim_k \widetilde{H}_1(X_{\prec b}, k) = 3$, we have $\beta_{2,abcd}(I) = 3$.

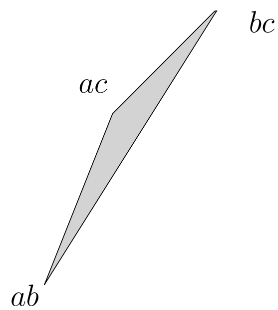

FIGURE 6. Subcomplex $X_{\preceq abc}$.

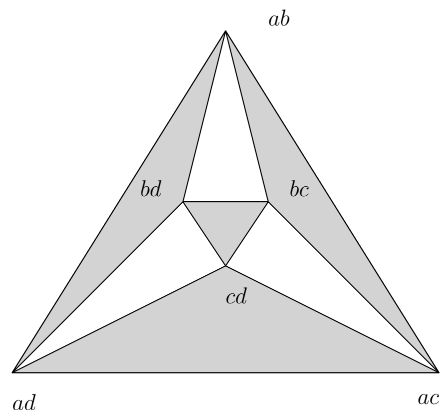

FIGURE 7. Subcomplex $X_{\prec abcd}$.

Remark 5.9. While we use the closed order ideal to test for cellular resolution, we use the open order ideal to calculate the Betti numbers.

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Theorem 5.10. *If $\mathcal{F}_X$ is a cellular resolution of I, then the $\mathcal{K}$-polynomial of I is the $\mathbb{N}^n$-graded Euler characteristic*

$$\mathcal{K}(S/I; X) = \sum_{F \in X} (-1)^{\dim F + 1} x^{a_F}.$$

<u>Examples of Cellular Resolutions</u>

(1) Planar Maps. (as we discussed in relation to the Buchberger graphs)

(2) Taylor Resolution: $X = \Delta_{r-1}$, the full $(r-1)$-simplex. This is "highly non-minimal."

(5) Minimal Triangulation of $\mathbb{RP}^2$.

$$0 \leftarrow S \leftarrow S^{10} \leftarrow S^{15} \leftarrow S^6 \leftarrow 0$$

is exact iff $\operatorname{char}(k) \neq 2$. The corresponding cell complex has 10 vertices, 15 edges, and 6 pentagonal faces (see p. 70 of Sturmfels-Miller for a diagram.)

(3) Permutohedron Ideals. The permutahedron is the convex hull of the action of the symmetric group. (Gorenstein)

Example 5.11. Take a solid hexagon, labeled with all permutations of the exponent vector $(1, 2, 3)$.

(4) Tree Ideals. They are defined as follows:

$$I = \left\langle \left(\prod_{i \in \sigma} x_i \right)^{n-|\sigma|+1} \middle| \, \emptyset \subseteq \sigma \subseteq [n] \right\rangle.$$

The tree ideals are Alexander dual to permutohedron ideals. They have $(n+1)^{n-1}$ standard monomials, one for each labeled tree on $n+1$ vertices. The Hilbert Series gives the parking functions = # reduced divisors on K_{n+1}.

These objects are important for Chip Firing (see "Monomials, Binomials, and Riemann-Roch" by Manjunath and Sturmfels).

The cell complex for $n = 3$ is presented in Figure 8.

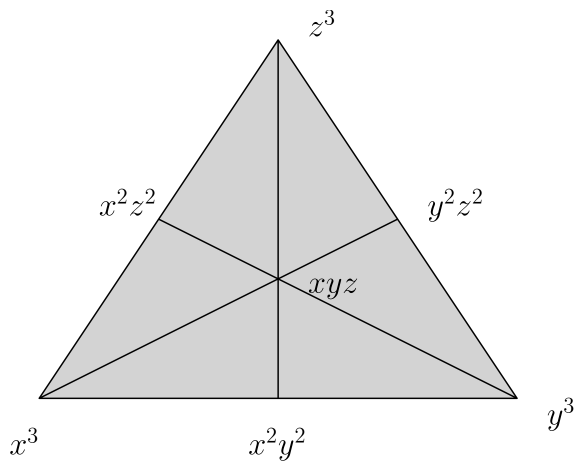


FIGURE 8. Tree ideal for $n = 3$.

(6) Simple Polytopes. Let $\mathcal{P}$ be a simple d-polytope with facets $F_1, \ldots, F_n$ and vertices $v_1, \ldots, v_r$. Label each vertex v_i by a squarefree monomial $\prod_{v_i \notin F_j} x_j$.

The corresponding ideal $I_{\mathcal{P}}$ is the irrelevant ideal in the toric Cox ring.

Exercise 5.12. Prove that $\mathcal{F}_{\mathcal{P}}$ is a linear minimal free resolution.

Question 5.13. Does every monomial ideal have a minimal cellular resolution?

No! (M. Velasco, Journal of Algebra 2008)

6. Friday, September 7, 2012

6.1. **The hull resolution.** For $t \in \mathbb{R}_{>0}$ and $a \in \mathbb{N}^n$ set $t^a = (t^{a_1}, t^{a_2}, \ldots, t^{a_n}) \in \mathbb{R}^n$. For a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, set

$$\mathcal{P}_t = \text{ conv } \{t^a : x^a \in I\}.$$

Lemma 6.1. *This is a convex polyhedron, specifically*

$$\mathcal{P}_t = \mathbb{R}^n_{\geq 0} \textit{ conv } \{t^a : x^a \in \textit{ min gens } I\}.$$

Proof. ($\subseteq$) Let $x^b \in I$. There is $x^a \in$ min gens(I) dividing $x^b \Rightarrow b \geq a \Rightarrow t^b - t^a \in \mathbb{R}^n_{\geq 0} \Rightarrow t^b \in$ RHS. Both are convex.

($\supseteq$) Need to show that for all $x^a \in$ min gens(I) : $t^a + \mathbb{R}^n_{\geq 0} \subseteq \mathcal{P}_t$. Consider $t^a + u$ where $u = (u_1, \ldots, u_n) \geq 0$.

Choose $r \in \mathbb{N}_+$ such that $0 \leq u_j \leq t^{a_j+r} - t^{a_j}$ for all j.

Let C be the convex hull of the 2^n points

$$t^a + \sum_{j \in J} (t^{a_j+r} - t^{a_j}) \cdot e_j$$

where $J \subseteq \{1, 2, \dots, n\}$. These represent the monomials $x^a(\prod_{j \in J} x_j)^r$.

The cube is contained in $\mathcal{P}_t$ and contains $t^a + u$. □

Proposition 6.2. *The face poset of $\mathcal{P}_t$ is independent of t for $t > (n+1)!$ The same holds for the subposet of bounded faces of $\mathcal{P}_t$.*

Definition 6.3. The hull complex hull(I) is the labeled polyhedral complex of all bounded faces of $\mathcal{P}_t$ for $t >> 0$.

Theorem 6.4. *The cellular free complex $\mathcal{F}_{hull}(I)$ is a resolution of S/I.*

Proof. By Proposition 4.5, NOS $X = (\mathrm{hull}(I))_{\leq \mathbf{b}}$ is acyclic over K

Claim: X is contractible. Set $v = t^{-\mathbf{b}}$. If t^a is a vertex of X then $a \leq b$, so

$$t^a \cdot v = t^{-b.-a} \leq t^{-b.b} = n.$$

All other monomials $x^c \in \min \mathrm{gens}(I)$ satisfy the hypothesis:

$$\exists i : c_i \geq b_i + 1 \Rightarrow t^c \cdot v = t^{-b} \cdot t^c \geq t^{c_i - b_i} \geq t > n.$$

The hyperplane $H = \{x \in \mathbb{R}^n | x \cdot v = n\}$ separates the vertices of X from all other vertices of $\mathcal{P}_t$.

$F = \mathcal{P}_t \cap H$ is a face of the polytope $Q = \mathcal{P}_t \cap H_{\geq 0}$ and X is the complex of faces of Q which are disjoint from F.

To complete the proof we cite Lemma 4.18: $X_{\leq b}$ is contractible. □

Corollary 6.5.

$$\mathcal{K}(S/I) = \chi(hull(I); x).$$

Example 6.6. hull(I) =

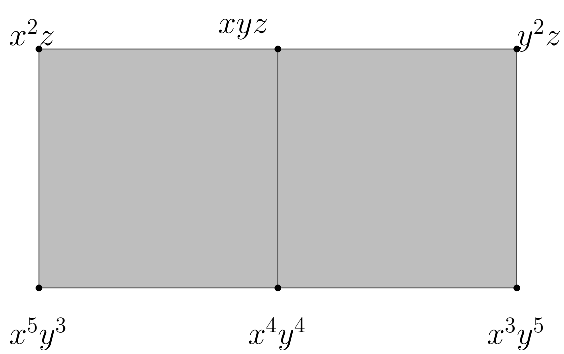


Figure 9. Hull Resolution.

6.2. **Subdividing the Simplex.** $J = \langle x_1^{d_1}, \dots, x_n^{d_n}, \dots \rangle$ is artinian. $\mathcal{P}_t$ has n special vertices $v_1, \dots, v_n$ that form an $(n-1)$-simplex.

Lemma 6.7. *The polytope $Q_t = conv(\{1\} \cup \Delta(J)) \cap \mathcal{P}_t$ is n-dimensional and has $\Delta(J)$ as a facet.*

Lemma 6.8. *Every bounded face of $\mathcal{P}_t$ survives as a face of Q_t.*

Theorem 6.9. *The hull complex $\Delta(J)$ is a (regular) polyhedral subdivision of $(n-1)$-simplex $\Delta(J)$. A face G of this complex lies in the boundary of hull(J) iff a_G fails to have full support.*

Exercise 6.10. Draw pictures illustrating Example 4.33. Take Q_t and hull(J) for

$$J = \langle x^5, y^5, z^5, x^3y^2, x^2y^3, x^3yz, x^2yz^2, xy^3z, xy^2z^2\rangle.$$

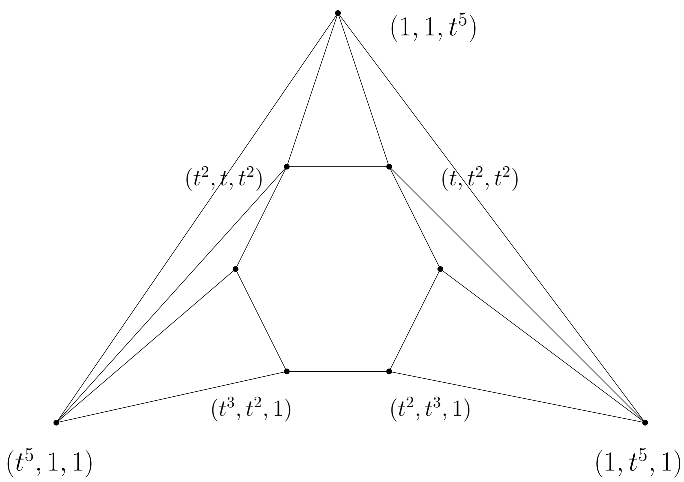


Figure 10. Q_t for the ideal J.

7. Monday, September 10, 2012

7.1. **Alexander Duality.** Minimal generators $x^\sigma = \prod_{i=\sigma} x_i$ become prime components $\mathfrak{m}^\sigma = \langle x_i \,|\, i \in \sigma\rangle$.

Proposition 7.1. *If I is a square free monomial ideal then $(I^*)^* = I$. Equivalently, $(\Delta^*)^* = \Delta$ for any simplicial complex Δ.*

Think about this in terms of *order ideals* and *filters* in a Boolean lattice on $n = 3$. In Figure 11, the white vertices are a filter, since the are closed upward, and the black vertices are an order ideal, since they are closed downward.

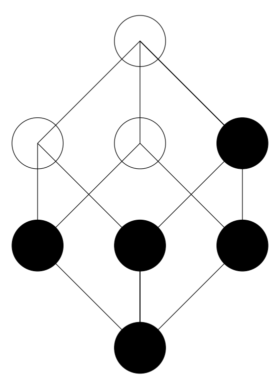


Figure 11. Boolean lattice of $2^[3]$.

Example 7.2. For $I_\Delta = \langle x_0x_1, y_0y_1, z_0z_1\rangle$. Then, the Alexander dual $I_{\Delta^*} = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \cap \langle z_0, z_1\rangle$. The complexes are represented in Figure 12.

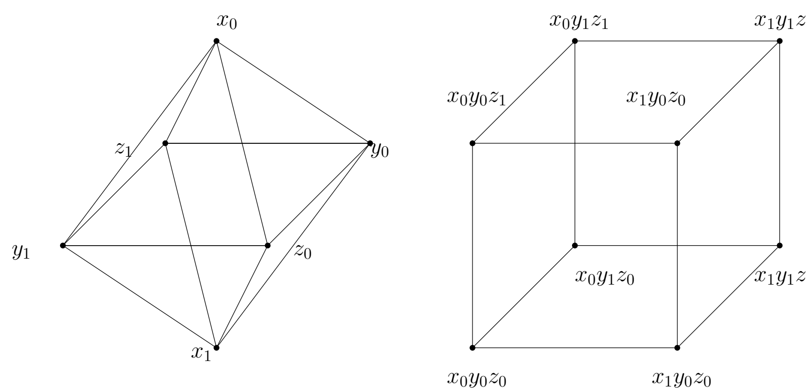


FIGURE 12. Δ and a Cellular Complex for Δ^*.

Theorem 7.3.
$$\widetilde{H}_{i-1}(\Delta^*, K) \cong \widetilde{H}^{n-2-i}(\Delta, K).$$

Theorem 7.4 (Alexander Inversion Formula)**.**
$$\mathcal{K}(S/I_\Delta, \bar{x}) = \mathcal{K}(I_{\Delta^*}, 1 - \bar{x}).$$

Proof. By Proposition 1.37, the Hilbert series of I_{Δ^*} is the sum of all monomials x^b divisible by $\prod_{j \notin \sigma} x_j$ for some face σ of Δ.
$$\Rightarrow formula$$
□

7.2. **Generators versus irreducible components.** A monomial ideal I in $S = K[x_1, \ldots, x_n]$ is irreducible if it has the form $\mathfrak{m}^b = \langle x_i^{b_i} \,|\, b_i \geq 1 \rangle$ for some $b \in \mathbb{N}^n$.

Example 7.5 (n= 3)**.** $\mathfrak{m}^{105} = \langle x, z^5 \rangle$.

Definition 7.6. An irreducible decomposition of I is
$$I = \mathfrak{m}^{b_1} \cap \mathfrak{m}^{b_2} \cap \cdots \cap \mathfrak{m}^{b_1}$$
If this is irredundant then the $\mathfrak{m}^{b_i}$ are irreducible components of I.

Lemma 7.7. *Every monomial ideal has an (irredundant) irreducible decomposition.*

Proof. If I is not irreducible then there exists generators $\mathfrak{m}, \mathfrak{m}'$ such that $\mathfrak{m}, \mathfrak{m}'$ are relatively prime. Now, $I = (I + \langle \mathfrak{m} \rangle) \cap (I + \langle \mathfrak{m}' \rangle)$. Iterate. □

Definition 7.8. If $a, b \in \mathbb{N}^n$ with $b \leq a$, let $a \setminus b$ denote the vector with coordinates
$$a_i \setminus b_i = \begin{cases} a_i + 1 - b_i & \text{if } b_i \geq 1 \\ 0 & \text{if } b_i = 0 \end{cases}.$$
For example,
$$(7, 6, 5) \setminus (2, 0, 3) = (6, 0, 3).$$

If all $\mathfrak{m}$ in generators of I divide x^a, then the Alexander dual of I with respect to x^a is $I^{[a]} = \bigcap \{\mathfrak{m}^{a \setminus b} \,|\, x^b \in \text{min gens}(I)\}$.
The canonical choice is $a = (1, 1, \ldots, 1)$.

Theorem 7.9.
$$(I^{[a]})^{[a]} = I.$$

Example 7.10. Let $I = \langle x^3, xy, yz^2 \rangle = \langle x^3, y \rangle \cap \langle x, z^3 \rangle$.
Take $a = (4,4,4)$.
Then $I^* = \langle x^2 \rangle \cap \langle x^4, y^4 \rangle \cap \langle y^4, z^3 \rangle = \langle x^2y^4, x^4z^3 \rangle$.

Theorem 7.11. *Assume all min gens of I divide x^a. Then I has a unique irredundant irreducible decomposition, namely*

$$I = \bigcap \{ \mathfrak{m}^{a \setminus b} \big| x^b \in \textit{min gens } (I^{[a]}) \}$$

Equivalently, the Alexander dual of I equals

$$I^{[a]} = x^{a \setminus b} \big| \mathfrak{m}^b \textit{ is an irreducible component of } I \rangle.$$

Remark 7.12. Computationally, finding a dual is much quicker than finding a decomposition in the usual way. So, this is a useful shortcut.

7.3. **Topological preparation for next time.** If X is a cell complex then its cochain complex $C^*(X, K)$ is the vector space dual of the chain complex $C_*(X, K)$.

If X' is a subcomplex of X then we have an inclusion $C_*(X', K) \subset C_*(X, K)$ and a surjection $C^*(X, K) \to C^*(X', K)$.

Definition 7.13. The relative cochain complex is defined by the exact sequence:

$$0 \to C^*(X, X'; K) \to C^*(X, K) \to C^*(X', K) \to 0.$$

It is the leftmost nonzero complex in the sequence. The i-th relative cohomology of the pair (X, X') is

$$H^i(X, X'; K) := H^i C^*(X, X'; K).$$

8. Wednesday, September 12, 2012

8.1. **Duality for Resolutions.**

Definition 8.1. For $a, b \in \mathbb{N}^n$ define $a \wedge b$ and $a \vee b$ in $\mathbb{N}^n$ by

$$x^{a \wedge b} = gcd(x^a, x^b), x^{a \vee b} = lcm(x^a, x^b) \Leftrightarrow (a \wedge b)_i = \min(a_i, b_i), (a \vee b)_I = \max(a_i, b_i).$$

Definition 8.2. A cell complex Y or a pair (Y, Y') is weakly colabeled if $a_G \geq a_F$ for faces $G \subset F$ and it is colabeled if $a_G = \bigvee_{F \text{ facet}, G \subset F} a_F$.

The cocellular monomial matrix supported on Y has the cochain complex $C^*(Y, K)$ for scalar entries, with faces of dimension $n-1$ in homological degree 0 and row and column labels a_G given by Y.

Example 8.3. The colabeled relative complex in Figure 13 is a minimal resolution for the permutohedron ideal.

Theorem 8.4 (Duality for Resolution)**.** *Fix a monomial Ideal I generated in degrees $\leq a$ and a length n cellular resolution $\mathcal{F}_X$ of $S/(I + m^{a+1})$ such that all face labels on X precede $a+1$. If $Y = a + 1 - X$, then $\mathcal{F}^Y$ is a weakly cocellular resolution of $(I^{[a]})_{\preceq a}$, and $\mathcal{F}^{Y_{\preceq a}}$ is a weakly cocellular resolution of the Alexander dual $I^{[a]}$. Both Y and $Y_{\preceq a}$ support minimal cocellular resolutions if $\mathcal{F}_X$ is minimal.*

Example 8.5. Apply this to the tree ideal I with minimal cellular resolution given by Figure 8. The minimal free resolution is:

$$0 \leftarrow S^7 \leftarrow S^{12} \leftarrow S^6 \leftarrow 0.$$

The first S^6 corresponds to triangles, S^{12} corresponds to edges, and S^7 to vertices. Subtracting each exponent vector in this labeling from $\mathbf{a} + 1 = (4,4,4)$ gives us the labeling in Figure 13.

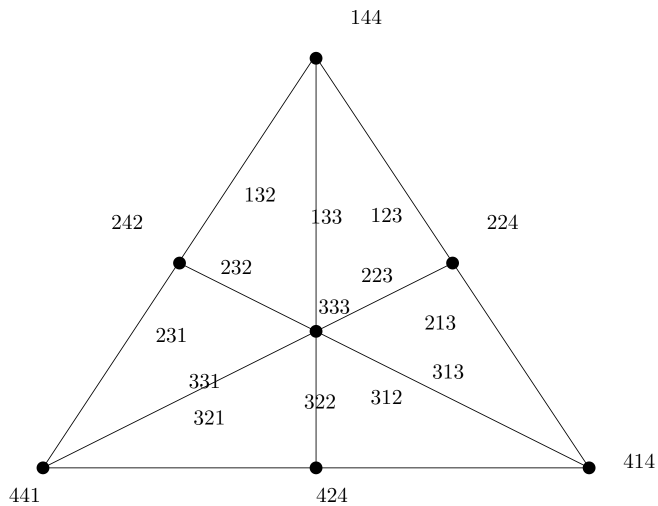


FIGURE 13. Colabeled Relative Complex.

We consider $\mathcal{F}^{Y_{\preceq a}}$, which essentially tells us to rip out the boundary, where we find the 4 coordinates.

Proof of Theorem 8.4. The proof uses the Nerve Lemma. If U is an acyclic cover of a polyhedral complex Y by acyclic polyhedral subcomplexes, then $\tilde{H}^i(Y, K) \cong \tilde{H}^i(\mathcal{N}(U), K)$. $\square$

8.2. **Cohull resolution and applications.**

Definition 8.6. Given an ideal I generated in degrees $\preceq a$, the cohull complex of I with respect to a is the weakly colabeled complex

$$cohull_a(I) = (a + 1 - X)_{\preceq a} \text{ for } X = hull(I^{[a]} + m^{a+1}).$$

Applying Theorem 5.37 to X we obtain:

Corollary 8.7. *$\mathcal{F}^{cohull_a(I)}$ is the free resolution of I.*

Remark 8.8 (Historical Note)**.** This construction came out of the desire to find the intersection of some set of ideals. It turned out to be easer to calculate the whole resolution, and read off the list of generators from that.

Theorem 8.9 (Duality for Betti numbers)**.** *If I is generated in degrees $\preceq a$ and $1 \preceq b \preceq a$, then*

$$\beta_{n-i,b}(S/I) = \beta_{i,a+1-b}(I^{[a]}).$$

"Alexander daulity for resolutions in 3 variables has a striking interpretation for planar graphs."

Theorem 8.10. *Let $I \supseteq m^a$ where $m = \langle x, y, z\rangle$. An axial planar map G supports a minimal cellular resolution of $K[x, y, z]/I$ if and only if its dual, the radial map $\tilde{G}$ supports a minimal cellular resolution of $K[x, y, z]/I^{[a]}$.*

Can you do this in four dimensions?

9. WEDNESDAY, SEPTEMBER 19. 2012

[Absent for Monday, September 17 due to Rosh Hashanah]

9.1. **Taylor Complexes and Genericity.** $I = \langle m_1, \ldots, m_r \rangle$ monomial ideal in $S = K[x_1, \ldots, x_n]$. For $\sigma \subset [r]$ set $m_\sigma = lcm(m_i : i \in \sigma)$ and $a_\sigma = \deg(m_\sigma)$.

Definition 9.1. Given any simplicial complex on Δ on $[r]$, the Taylor complex $\mathcal{F}_\Delta$ is the sequence of monomial matrices from the boundary map of Δ with face labels m_σ.

Lemma 9.2. *$\mathcal{F}_\Delta$ is a resolution iff for every monomial m, the subcomplex $\Delta_{\leq m} = \{\sigma \in \Delta | m_\sigma$ divides $m\}$ is acyclic (no homology) over K.*

Lemma 9.3. *$\mathcal{F}_\Delta$ is minimal iff $\forall \sigma \in \Delta$ and $\forall i \in \sigma$, $m_\sigma \neq m_{\sigma \setminus i}$.*

Definition 9.4. Call I generic if whenever two generators m_i and m_j have the same degree $\deg_{x_k}(m_i) = \deg_{x_k}(m_j)$. Then there exists m_l that strictly divides $m_{ij} = lcm(m_i, m_j)$ in every variable.

Remark 9.5. Strongly generic, in the sense of Chapter 3, implies generic. However, the reverse implication is not true.

9.2. **The Scarf Complex.**

Definition 9.6. The Scarf Complex Δ_I is the collection of subsets of $\{m_1, \ldots, m_r\}$ whose lcm is unique. Explicitly,

$$\Delta_I = \{\sigma \subseteq [r] | \forall \tau \subseteq [r], m_\sigma = m_\tau \Rightarrow \sigma = \tau.\}.$$

Lemma 9.7. *The Scarf complex Δ_I is a simplicial complex. Its dimension is at most $n-1$.*

Example 9.8. For the ideal, $\langle x^2z, xy, y^2z, z^2 \rangle$, the scarf complex is:

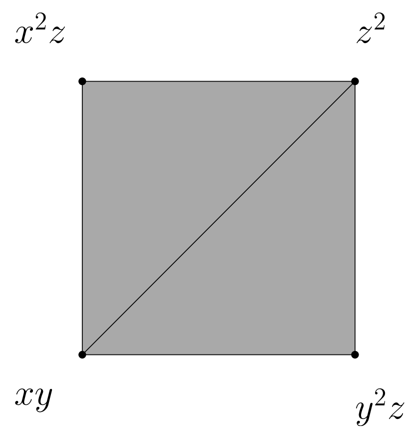


FIGURE 14. Scarf Complex.

with free resolution $0 \leftarrow S^4 \leftarrow S^5 \leftarrow S^2 \leftarrow 0$.

Example 9.9. $I = \langle xy, xz, yz \rangle$, then Δ_I = three isolated vertices. $\mathcal{F}_{\Delta_I} : 0 \leftarrow S \leftarrow S^3 \leftarrow 0$ is not exact.

$I = \langle x, y, z \rangle, \Delta_I$ = solid triangle, $\mathcal{F}_{\Delta_I}$ is the Koszul complex, which is exact.

Call $\mathcal{F}_{\Delta_I}$ the algebraic Scarf Complex of I.

Proposition 9.10. *Every free resolution of S/I contains $\mathcal{F}_{\Delta_I}$ as a subcomplex. (The monomial matrices contain it as a block)*

Theorem 9.11. *If I is any monomial ideal then its scarf complex is a sub complex of hull(I). If I is generic then Δ_I = hull(I), so its algebraic Scarf complex $\mathcal{F}_{\Delta_I}$ minimally resolves the quotient.*

Therefore, we have a nice upper bound for a resolution from the hull complex, and a nice lower bound from the scarf complex, and they agree in the generic case.

Corollary 9.12. *The minimal free resolution of a generic monomial ideal is independent of char(K). The total Betti number $\beta_i(I) = \sum_{a\in\mathbb{N}^n}\beta_{a,i}$ equals the number $f_i(\Delta)$ of i-dimensional faces in the scarf complex.*

Corollary 9.13. $\mathcal{K}(S/I) = \sum_{\sigma\in\Delta_I}(-1)^{|\sigma|}m_\sigma$.

How does Alexander duality interact with the scarf complex?

Definition 9.14. For $b, u \in \mathbb{N}^n$, set $\hat{b} = \sum_{i:b_i\leq u_i} b_i e_i$.

Corollary 9.15. *Given I is generic, fix $u \in \mathbb{N}^n$ such that every minimal generator of I divides x^4. Set $I' = I + m^{u+1}$. The irredundant irreducible decomposition of I is*

$$I = \bigcap_{G\in\Delta_{I'}} m^{\hat{a}_G}.$$

This allows you to create the following irreducible decomposition:

Example 9.16. Specifically, use $u = (3, 3, 3)$.

$$\begin{aligned} I &= \langle x^3y^2z, x^2yz^3, xy^3z^2\rangle \\ &= \langle x\rangle \cap \langle x\rangle \cap \langle y\rangle \cap \langle z\rangle \cap \langle x^2, y^3\rangle \cap \langle y^2, z^3\rangle \cap \langle z^2, x^3\rangle \cap \langle x^3, y^3, z^3\rangle. \end{aligned}$$

Theorem 9.17. *A generic monomial ideal I is Cohen-Macaulay if and only if all irreducible components of I have the same dimension. More generally, the projective dimension of S/I equals the maximal number of generators of an irreducible component.*

Alternatively, the number of yellow vertices and white vertices are the same.

10. Friday, September 21, 2012

10.1. **Genericity by Deformation.** A deformation fo a monomial ideal $I = \langle m_1, m_2, \ldots, m_r\rangle$ is a choice of vectors $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{in})$ for $i \in [r]$ satisfying

$$a_{is} < a_{js} \Rightarrow a_{is} + \varepsilon_{is} < a_{js} + \varepsilon_{js}$$

and $a_{is} = 0 \Rightarrow \varepsilon_{is} = 0$.

Formally introudce (in a suitable polynomial ring):

$$I_\varepsilon = \langle m_1x^{\varepsilon_1}, \ldots, m_rx^{\varepsilon_r}\rangle = \langle x^{a_1+\varepsilon_1}, \ldots, x^{a_r+\varepsilon_r}\rangle.$$

Theorem 10.1. *Let I_ε be a generic deformation of I. Define Δ_I^ε by labeling each face of the Scarf complex Δ_{I_ε} by m_σ instead of lcm$(m_ix^{\varepsilon_i} : i \in \sigma)$. The resulting Taylor complex $\mathcal{F}_{\Delta_I^\varepsilon}$ resolves I.*

Example 10.2. The images below depict an ideal before and after deformation.

Theorem 10.3. *Suppose* $\min(I) \preceq u$ *and set $I^* = I + \mathfrak{m}^{u+1}$. The following are equivalent:*

1. *I is generic.*
2. *$\mathcal{F}_{\Delta^*}$ is a minimal free resolution of S/I^*.*
3. *$\Delta_{I^*} = hull(I^*)$.*
4. *$I = \cap\{\mathfrak{m}^{\hat{a_\sigma}} | \sigma \in \Delta_{I^*}$ and $|\sigma| = n\}$ is the irredundant, irreducible decomposition of I.*
5. *For each irreducible component $\mathfrak{m}^b$ of I^*, there exists a face of Δ_{I^*} labeled by $a_\sigma = b$.*
6. *$\mathcal{F}_{\Delta_I}$ is a free resolution of S/I and no variable x_k appears with the same non-zero exponent in m_i and m_j for any edge $\{i, j\}$ of Δ_I.*
7. *If $\sigma \notin \Delta_{I^*}$, then some monomial $m \in I$ divides m_σ.*
8. *The Scarf complex Δ_{I^*} is unchanged by arbitrary deformations of I^*.*

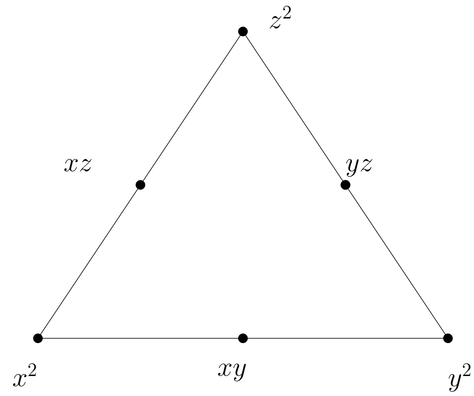


FIGURE 15. Before Deformation.

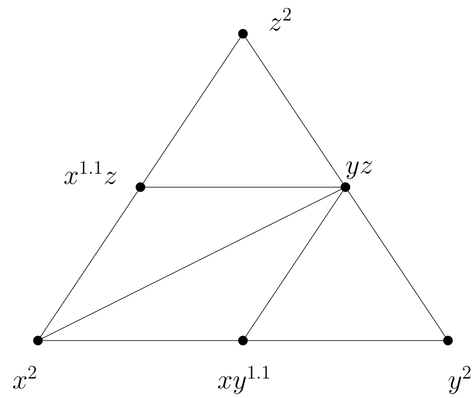


FIGURE 16. After Deformation.

10.2. **Bounds on Betti Numbers.**

Definition 10.4. The cyclic polytope $C_n(r) = \text{conv}\{(t_i, t_i^2, \ldots, t_i^n) : i = 1, 2, \ldots, r\}$. (David Gale)

Theorem 10.5 (Ziegler, Theorem 8.23)**.** *The Upper Bound Theorem for Convex Polytopes: The number $\beta_i(I)$ of minimal i-th syzygies of any monomial ideal I with r generators in n variables is $\leq \#C_{i,n,r}$ of i-dimensional faces of $C_n(r)$. If $i = n-1$ then even $\beta_i(I) \leq C_{n-1,n,r} - 1$.*

Example 10.6. $n = 3, S = k[x, y, z]$ planar graphs.

$$C_{1,3,r} = 3r - 6, \qquad C_{2,3,r} = 2r - 4.$$

Example 10.7. $n = 4, S = k[a, b, c, d]$. Cyclic polytopes are "neighborly": $C_{1,4,r} = \binom{r}{2}$, i.e. every pair of vertices is connected by an edge.

$$C_{0,4,r} = r, \quad C_{1,4,r} = \frac{1}{2}r(r-1), \quad C_{2,4,r} = r(r-3), \quad C_{3,4,r} = \frac{1}{2}r(r-3).$$

Example 10.8. $(r = 12)$

$$I = \langle a^9, b^9, c^9, d^9, a^6b^7c^4d, a^2b^3c^8d^5, a^5b^8c^3d^2, ab^4c^7d^6, a^8b^5c^2d^3, a^4bc^6d^7, a^7b^6cd^4, a^3b^2c^5d^8 \rangle.$$

Every pairwise first syzygy is minimal. The resolution is:

$$0 \leftarrow I \leftarrow S^{12} \leftarrow S^{66} \leftarrow S^{108} \leftarrow S^{53} \leftarrow 0.$$

Question 10.9. Are these bounds tight?

No!

Theorem 10.10 (Hosten & Morris). *Let HM_n be the number of simplicial complexes on $\{1, 2, \ldots, n-1\}$ (equal to the number of anti-chains in the Boolean lattice) such that no pair of faces covers a ll of $\{1, \ldots, n-1\}$. Then the maximum number of generators of a neighborly monomial ideal in n variables equals HM_n.*

n	3	4	5	6	7	8
HM_n	4	12	81	2646	1,422,564	229,809,982,112

11. Monday, September 24, 2012

(Guest lecture by Thomas Kahle.)

Part I was about monomial ideals in $S = k[x_1, \ldots, x_n] = k[\mathbb{N}^n]$.
S has a monoid of exponents $\mathbb{N}^n = \text{posHull}\{e_1, \ldots, e_n\}$, where e_i is the i-th generators of $\mathbb{Z}^n$.

We generalize to Q, the positive Hull of $a_1, \ldots, a_n \in A$ some abelian group. Then $Q = \{\sum_{i=1}^n n_i a_i, n_i \in \mathbb{N}\}$.
Convention: Q is finitely generated, cancellative (i.e. $a + c = b + c \Rightarrow a = b$), and Q has identity.

Definition 11.1. The semigroup ring ("monoid algebra") is the k-algebra with k-basis $\{t^a : a \in Q\}$ and multiplication $t^a t^b = t^{a+b}$. (k could be a ring) This algebra is denoted $K[Q]$

Definition 11.2. λt^a is a monomial. $t^a - \lambda t^b$ is a binomial. A monomial (resp. binomial) ideal is an ideal generated by monomials (resp. binomials).

Question 11.3. How do we compute with $k[Q]$?

We give a presentation for Q:

$$A \leftarrow \mathbb{Z}^n \leftarrow L \leftarrow 0.$$

(L is an integer lattice, i.e. subgroup of $\mathbb{Z}^n$, and $e_i \in \mathbb{Z}^n \mapsto a_i \in A$)
So, $Q = \mathbb{N}^n / \sim_L$, where $u \sim_L v$ iff $u - v \in L$.

Definition 11.4. Given an integer lattice L, the lattice ideal I_L is the binomial ideal $I_L = \langle x^u - x^v : u, v \in \mathbb{N}^n, u - v \in L\rangle \subseteq S = k[x_1, \ldots, x_n]$.

Theorem 11.5. *$k[Q] \cong S/I_L$, and $deg(t^a) = a$.*

Remark 11.6. The Q-graded Hilbert function takes only values in $\{0, 1\}$ on S/I_L.

We consider some Lattices in $\mathbb{Z}^3$:

Example 11.7.

$$L = \mathbb{Z}(3, 4, -5), \qquad A = \mathbb{Z}^2, \qquad Q = \mathbb{N}\{\binom{5}{2}, \binom{0}{1}, \binom{3}{2}\}$$

I_L is minimally generated as $\langle x^3y^4 - z^5\rangle$.

Example 11.8.

$$Q = \mathbb{N}\{3, 4, 5\}, \qquad A = \mathbb{Z}, \qquad L = \{(u, v, w) \in \mathbb{N}^3 : 3u + 4v + 5w = 0\}$$

In this case, I_L is generated as $\langle x^3 - yz, x^2y - z^2, xz - y^2\rangle$.

Example 11.9.

$$L = \{(u, v, w) \in \mathbb{Z}^3, u + v + w \in 2\mathbb{Z}.\}$$

Then $2e_i \in L$ for each i. Here,

$$I_L = \langle x^2 - 1, xy - 1, yz - 1\rangle = \langle x - 1, y - 1, z - 1\rangle \cap \langle x + 1, y + 1, z + 1\rangle.$$

$k[Q]$ is not an integral domain.

Theorem 11.10. *The following are equivalent:*

1. *I_L is prime.*
2. *$k[Q]$ is an integral domain.*
3. *The group generated by Q inside A is free abelian.*
4. *Q is an affine semigroup, i.e. a finitely generated subsemigroup of $\mathbb{Z}^d$, for some d.*

Proposition 11.11.

$$\dim k[Q] = n - \ \mathit{rank}(L).$$

11.1. **Polyhedral Cones and Affine Semigroups.** Let Q be an affine semigroup.

$$\mathbb{Z}^d \supseteq A \xleftarrow{A} \mathbb{Z}^n \leftarrow L \leftarrow 0.$$

Def. of polyhedral cone.

Definition 11.12. $T \subseteq Q$ is an ideal if $T - Q \subseteq T$. A subsemigroup $F \subseteq Q$ is a face if $Q \setminus F$ is an ideal. Q is pointed if 0 is the only unit.

Lemma 11.13. *$F \subseteq Q$ is a face of Q if and only if $Q \setminus F$ is a prime ideal p_F.*

Lemma 11.14. *Faces of Q are $1 - 1$ with faces of cone(A).*

12. Friday, October 5, 2012

[Class on Monday and Wednesday missed for Sukkot]
Usually deg is surjective, so we can write

$$0 \to L \to \mathbb{Z}^n \overset{\deg}{\to} A \to 0$$

and the grading of S is specified by the sub lattice L of $\mathbb{Z}^n$.

Example 12.1. $n = 2, S = K[x, y]$.

(1) $L = \text{span}\{\binom{2}{0}, \binom{1}{1}\}$; $A \cong \mathbb{Z}_2$. $S = S_{\text{even}} \oplus S_{\text{odd}}$. This is the super- polynomial ring.
(2) $L = \text{span}\{\binom{5}{-3}\} \Rightarrow A = \cong \mathbb{Z}$, $\deg(x) = 3$, $\deg(y) = 5$.

$$S = S_0 \oplus S_3 \oplus S_5 \oplus S_6 \oplus S_8 \oplus S_9 \oplus \cdots$$

The Hilbert Series is

$$\frac{1}{(1 - t^3)(1 - t^5)} = \sum \dim_K(S_n)t^n.$$

In general $S = \bigoplus_{a \in A} S_a$ and $S_0 = K[L \cap \mathbb{N}^n]$ is a normal semigroup ring.

Proposition 12.2. *Each S_a is a finitely-generated S_0-module.*

(Missed a few notes here.) This is related to Spec(S_0).

Theorem 12.3. *Let S be multi graded by A and $Q = \mathit{deg}(\mathbb{N}^n)$. The following are equivalent:*

1. *There exists $a \in A$, such that S_a is finite-dimensional over K.*
2. *The only polynomials of degree zero are constants, i.e. $S_0 \cong K$.*
3. *For all $a \in A$, S_a is finite-dimensional over K.*

(4) For all finitely-generated A-graded S-modules M, and for all $a \in A$, M_a is finite-dimensional over K.
(5) Zero is the only non-negative vector in the lattice L. That is $L \cap \mathbb{N}^n = \{0\}$.
(6) The semigroup Q has no units, and no variable x_i has degree 0.

If this holds, the grading is positive.

Positive gradings can be given by $\mathbb{Z}^n \to A \subset \mathbb{Z}^d$ with $Q = \deg(\mathbb{Z}^n) \subset \mathbb{N}^d$.
Helpful for Review: Modules over PIDs, Smith Normal Form.

Proposition 12.4. *Let S be multigraded by a torsion-free abelian group A. All associated primes of multigraded S-modules are homogeneous.*

Example 12.5. $S = K[x,y], A = \mathbb{Z}$, $\deg(x) = 1$, $\deg(y) = -1$.

$$I = \langle x^3y + x^2, xy^2 + y\rangle = \langle xy + 1\rangle \cap \langle x^2, y\rangle.$$

The ideal is homogeneous, as well as its associated primes.

Remark 12.6. How do you know if an ideal is homogeneous? Calculate the reduced Gröbner basis; every term should be homogeneous.

In a super algebra, $\langle x^2 - 1\rangle$, is homogenous of degree zero. However, if you break it up into primes $\langle x^2 - 1\rangle = \langle x + 1\rangle \cap \langle x - 1\rangle$.

12.1. **Hilbert Series and K-polynomials.** Suppose S is positively graded as described above. For any finitely-generated, graded S-module, the Hilbert series

$$H(M,t) = \sum_{a \in A} \dim_K(M_a) t^a$$

lives in $\mathbb{Z}[[A]] = \prod_{a \in A} \mathbb{Z}t^a$, an additive group.

This group contains the ring $\mathbb{Z}[[Q]][A] = \mathbb{Z}[[Q]] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[A]$, the completion of the semigroup algebra tensor the Laurent Series.

Lemma 12.7.

$$H(S,t) = \frac{1}{(1-t^{a_1})(1-t^{a_2})\cdots(1-t^{a_n})}$$

holds in $\mathbb{Z}[[Q]]$.

Theorem 12.8. *For every finitely-generated graded S-module M, there exists a unique Laurent polynomial $K(M;t) \in \mathbb{Z}[A]$ such that*

$$H(M,t) = \frac{K(M;t)}{\prod_{i=1}^n (1-t^{a_i})}$$

holds in $\mathbb{Z}[[Q]][A]$. This is the K-polynomial.

We have a finite minimal free resolution. Every free summand is multigraded, so everything will be given by a monomial matrix. We can do inclusion-exclusion, and this will give the K-polynomial, as well.

Easier strategy: Pass to an initial ideal. The K-polynomial will be the same.

Sneak Peek to next week: Given a variety $X \subset \mathbb{P}^n$, we have notions of $\dim(X)$ and degree(X). We will have multidegree versions of both notions, which will be obtained from the polynomial in t.

13. Wednesday, October 10

13.1. **Syzygies of Lattice Ideals.** Let $Q = \mathbb{N}\{a_1, \ldots, a_n\} \subset \mathbb{Z}^d = A$ be a pointed affine semigroup. Let $S = K[x_1, \ldots, x_n]$ be a positively graded ring. $K[Q] = S/I_L$ has a Hilbert series

$$\sum_{b \in Q} t^b = \frac{\mathcal{K}_Q(t_1, \ldots, t_d)}{(1 - t^{a_1}) \cdots (1 - t^{a_n})}$$

and this derives from the minimal free resolution

$$0 \leftarrow I_L \leftarrow \bigoplus_{b \in Q} S(-b)^{\beta_{0,b}} \leftarrow \cdots \leftarrow \bigoplus_{b \in Q} S(-b)^{\beta_{r,b}} \leftarrow 0,$$

where we define $\beta_{i,b} = \#$ minimal i-th syzygies of $K[Q]$ in degree b. Fact: $r \leq n-1$ by Hilbert's syzygy theorem.

Example 13.1 (Twisted Cubic Curve)**.** $n = 4, d = 2$. The semigroup $Q = \mathbb{N}\{\binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{1}{3}\}$.

$$0 \leftarrow I_L \overset{[bd-c^2, bc-ad, ac-b^2]}{\longleftarrow} \overset{S(-(2,4))}{\oplus S(-(2,3))} \oplus S(-(2,2)) \overset{\begin{bmatrix} a & b \\ b & c \\ c & d \end{bmatrix}}{\longleftarrow} \overset{S(-(3,4))}{\oplus S(-(3,5))} \leftarrow 0.$$

[Hilbert-Burch,Cohen-Macaulay of codimension 2]

The Hilbert series:

$$\frac{\mathcal{K}_Q(s,t)}{(1-s)(1-st)(1-st^2)(1-st^3)} = \frac{1 + st + st^2}{(1-s)(1-st^3)}.$$

The multi degree is a degree 2 homogeneous polynomial, specifically:

$$3s^2 + 9st + 6t^2 = (s + 2t)(s + 3t) + s(s + 3t) + s(s + 3t) + s(s + t).$$

This latter expression would be obtained by considering the initial ideal as the intersection $\langle c, d\rangle \cap \langle a, d\rangle \cap \langle a, b\rangle$.

For $b \in Q$ consider the simplicial complex $\sum u_i a_i = b$, then

$$\Delta_b = \{I \subseteq \{1, 2, \ldots, n\} | b - \sum_{i \in I} a_i \in Q\}.$$

$=$ simplicial complex generated by $\{\text{supp}(x^u) | x^u \in S_b\}$.

Theorem 13.2.

$$\beta_{j,b} = \dim_K(\tilde{H}_j(\Delta_b, K)).$$

Idea of Proof. Start with $\beta_{j,b} = \dim_K(\text{Tor}_S^{j+1}(K, K[Q])_b)$. Tensor the Koszul complex $\mathbf{K}_0$ with $K[Q]$. In degree b this is the reduced chain complex of Δ_b. □

Example 13.3 (Twisted Cubic, Part 2)**.** Take $b = \binom{3}{4}$. Then,

$$S_{\binom{3}{4}} = K\{and, ac^2, b^2c\}.$$

The simplicial complex $\Delta_{\binom{3}{4}}$ is given in Figure 17.

Corollary 13.4. *I_L has a minimal generator in degree b iff Δ_b is disconnected.*

We want to beat Hilbert's Syzygy Theorem for $n >> n - d$ (Lots of generators in low degree). (Recall the upper bound from HST is $r \leq n - 1$.

Corollary 13.5. *The projective dimension of $I_L \leq 2^{n-d} - 2$.*

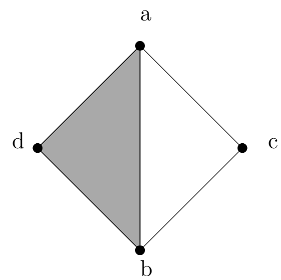


Figure 17. Simplicial Complex for $\binom{3}{4}$.

Proof. Let $F_1, \dots, F_s$ be the facets of Δ_b. There exist monomials $x^{u_1}, \dots, x^{u_s}$ of degree b with $\mathrm{supp}(x^{u_i}) = F_i$.

Claim: $s \le 2^{n-d}$. Otherwise, $\exists i \neq j$ such that $u_i \equiv u_j \mod 2$. The midpoint $\frac{1}{2}(u_i + u_j)$ is in $\mathbb{N}^n$ and has degree b, and its support $F_i \cup F_j$ strictly contains F_i. □

This bound is actually tight.

Example 13.6. Let $n = 8, d = 5 \Rightarrow n - d = 3$. Take the lattice

$$L = \mathrm{span}\begin{bmatrix} 1 & 1 & 1 & 2 & -2 & -1 & -1 & -1 \\ 1 & 2 & -2 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 2 & -2 \end{bmatrix}.$$

What are $a_1, \dots, a_8 \in \mathbb{Z}^5$? We have the following free resolution:

$$0 \leftarrow I_L \leftarrow S^{13} \leftarrow S^{44} \leftarrow S^{67} \leftarrow S^{56} \leftarrow S^{28} \leftarrow S^8 \leftarrow S^1 \leftarrow 0.$$

There exists a special $b \in \mathbb{N}^8/L = Q$ such that $\Delta_b =$ boundary of the 7-simplex.

13.2. **Fast Forward.** How do we build resolutions?

Corollary 13.7 (9.25 in the Text, Page 188)**.** *The minimal free resolution of a generic lattice ideal is its Scarf complex.*

We only defined a Scarf complex for a monomial ideal.

Definition 13.8. The Scarf complex of a generic lattice ideal is the image modulo the L-action of the Scarf complex of the monomial module

$$M_L = S\{x^u : u \in L\}.$$

Example 13.9. Consider the lattice $L = \mathbb{Z}\binom{1}{-1} \subset \mathbb{Z}^2$. Then, we draw the lattice and label as in Figure 18.

An image in 3 dimensions is found on page 185 of the text.

14. Guest Lecture by Dave Bayer, October 11, 2012

For a long time, developments in monomial ideals mirrored those in toric ideals.

Example 14.1. Let $I = \langle b^2 - ac, bc - ad, c^2 - bd\rangle$. The location of a generator in a lattice gives a simplicial complex that relates to the ideal. The lattice corresponding to our ideal is depicted in Figure 19.

Let $k[a, b]/(b^2 - a^2)$. To get a standard monomial, send $b^2 \to a^2$, as pictured in Figure 20. Then, $L : \langle(-2, 2)\rangle$.

We can draw a picture of a staircase which represents this notion.

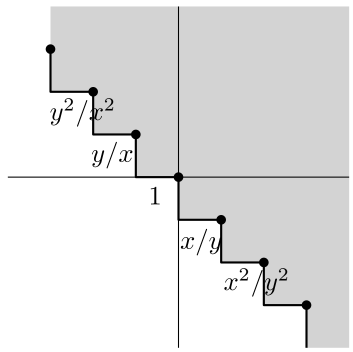


FIGURE 18. Lattice with Labeling for Scarf Complex.

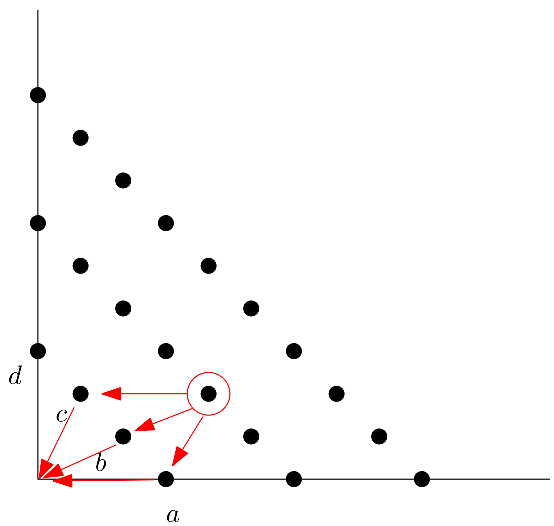


FIGURE 19. Lattice for $k[a, b, c]/I$, with Generator and Associated Complex.

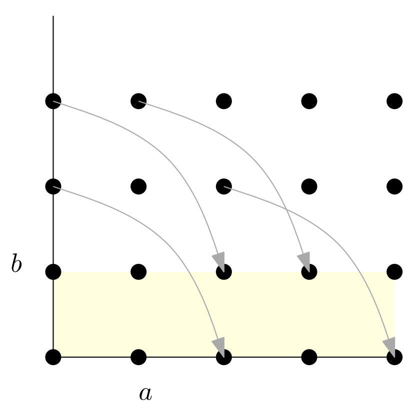


FIGURE 20. Lattice for $k[a, b]/(b^2 - a^2)$.

Definition 14.2. Laurent monomial module. Previously when working with $k[x_1, \ldots, x_n]$, we dealt with $k[\mathbb{N}^n]$. Now we are working in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $L \subset \mathbb{Z}^n$be a lattice. Let $J = \langle x^u | u \in L\rangle$.

Example 14.3. Consider $k[a,b]/(b-a)$. The corresponding lattice is $L = \langle(1,-1)\rangle \subset \mathbb{Z}^2$, as pictured in Figure 21.

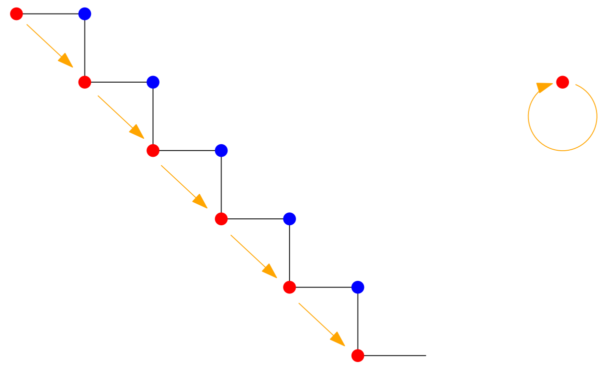


FIGURE 21. Lattice for $k[a,b]/(b-a)$ and Corresponding Cellular Visualization.

Example 14.4. Consider the following lattice in $\mathbb{Z}^4$:

$$L = \langle(-1, 2-1, 0), (-1, 1, 1, -1), (0, -1, 2, -1)\rangle.$$

Though this is in four dimensions, we only care about the two-dimensional slice in which the lattice lies. This is a grid, similar to the image in Figure 22.

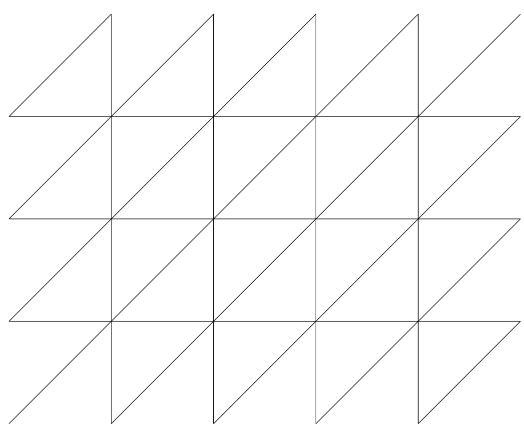


FIGURE 22. Grid Associated to L.

A cell from this grid can be labeled as in Figure 23.

This picture gives us the first minimal syzygy of the twisted cubic.

15. Monday, October 15, 2012

15.1. Free Resolutions of Lattice Ideals.

Definition 15.1. Let $L \subset \mathbb{Z}^n$ be a lattice, such that $L \cap \mathbb{N}^n = \{0\}$. The group algebra of L over $S = K[x_1, \ldots, x_n]$ is

$$S[L] = K\{x^u z^v : u \in \mathbb{N}^n, v \in L\}.$$

It is $\mathbb{Z}^n$-graded via $\deg(x^u z^v) = u + v$. The lattice module M_L is a $\mathbb{Z}^n$-graded $S[L]$-module

$$M_L \cong S[L]/\langle x^u - x^v z^{u-v} : u, v \in \mathbb{N}^n, u - v \in L\rangle.$$

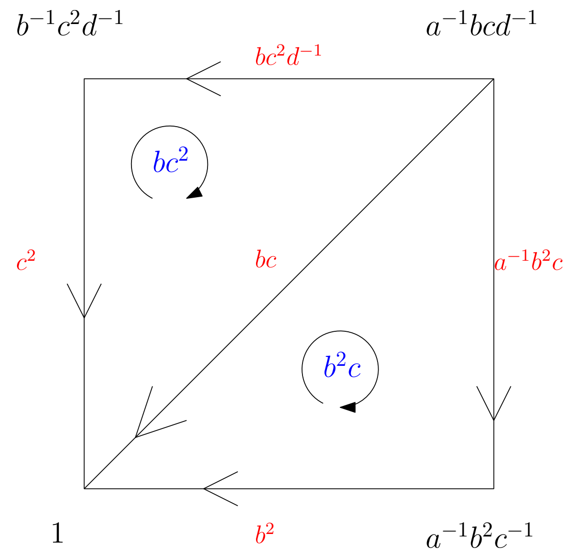


FIGURE 23. Cell Complex Associated to L.

Consider two categories:

$$\mathscr{A} = \{\mathbb{Z}^n\text{-graded f.g. } S[L]\text{-modules}\}.$$

$$\mathscr{B} = \{\mathbb{Z}^n/L\text{-graded f.g. } S\text{-modules}\}.$$

and the functor $\pi : \mathscr{A} \to \mathscr{B}$, sending $M \mapsto M \otimes_{S[L]} S$. Set all z's to 1. As an $S[L]$-module, S can be written as

$$S = S[L]/\langle z^v - 1 : v \in L\rangle.$$

The ideal we quotient by is called the "augmentation ideal."

Key observation: $\pi(M_L) = S/I_L$.

Theorem 15.2. *π is an equivalence of categories.*

Corollary 15.3. *If $\mathcal{F}_\bullet$ is any $\mathbb{Z}^n$-graded free resolution of M_L over $S[L]$, then $\pi(\mathcal{F}_\bullet)$ is a $\mathbb{Z}^n/L$-graded free resolution of S/I_L over S.*

Moreover, $\mathcal{F}_\bullet$ is minimal iff $\pi(\mathcal{F}_\bullet)$ is minimal iff $\pi(\mathcal{F}_\bullet)$ is minimal.

A resolution of M_L over $S[L]$ is a resolution of M_L as an S-module with a free L-action. Main example: Hull Resolution.

$$\text{hull}(M_L) : 0 \leftarrow S[L] \leftarrow S[L]^{\beta_1} \leftarrow S[L]^{\beta_2} \leftarrow \cdots \leftarrow 0,$$

where $\beta_i = \#L$-equivalence classes of i-faces of $\text{hull}(M_L)$.

Definition 15.4. The hull resolution of $K[Q] = S/I_L$ is $\pi(\text{hull}(M_L))$. It is obtained by setting $z = 1$ in all maps.

Example 15.5 (Steven Karp 11/21)**.** Let L be unimodular; i.e. every linearly independent d-subset of $\{a_1, \ldots, a_n\}$ is a basis of $\mathbb{Z}^d$.

Consider the Lawrence lifting $\Lambda(L) = \{(u, -u) \in \mathbb{Z}^{2n} | u \in L\}$ and its ideal $I_{\Lambda(L)} = \langle x^u y^v - x^u y^v | u - v \in L\rangle$ in $2n$ variables.

15.2. **Genericity and the Scarf Complex.** A Laurent monomial module M in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is generic if all its minimal first syzygies $x^u e_i - x^v e_j$ have full support.

Theorem 15.6. *For a generic M, the following are identical:*

(1) The Scarf complex of M.
(2) The hull resolution of M.
(3) The minimal free resolution of M.

Call a lattice L generic if M_L is generic.

Example 15.7. Let $n = 4$. Let $L = \ker[20\ 24\ 25\ 31] \subset \mathbb{Z}^4$. I_L is the ideal of the curve $(t^{20}, t^{24}, t^{25}, t^{31})$.

$$S[L] = K[a, b, c, d][z^v : v \in L].$$
$$M_L = S[L]/\langle a^4 - bcdz^{(4,-1,-1,-1)}, \ldots, c^3 - abdz^{(-1,-1,-1,3)}\rangle.$$

The quotiented ideal is generated by seven relations, each of full support.

By the theorem, hull = Scarf = minimal resolution of S/I_L

$$0 \leftarrow S \leftarrow S^7 \leftarrow S^{12} \leftarrow S^6 \leftarrow 0.$$

Up to the L-action, there are 6 tetrahedra, 12 triangles, 7 edges and 1 vertex.

In the 2d lattice case, you can learn about the initial ideal by looking at the link of the lattice, as pictured in Figure 24.

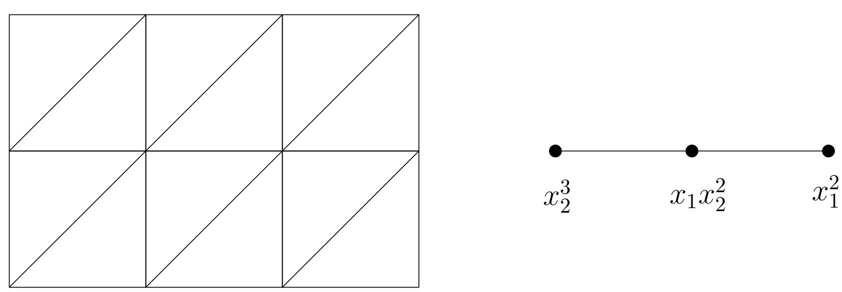


FIGURE 24. Link of the Lattice and its Initial Ideal.

We can similarly take the link of the 3d lattice, which looks like Figure 25.

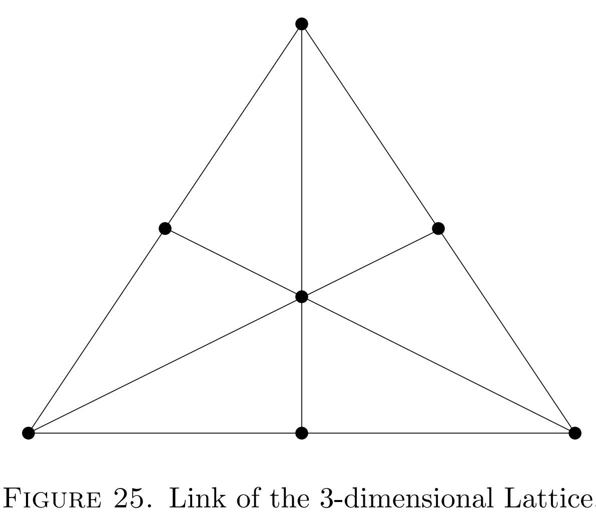


FIGURE 25. Link of the 3-dimensional Lattice.

Reference: "Generic Lattice Ideals". Quote from p. 189: "The deterministic construction of generic lattices with prescribed properties is an open problem ... difficult..."

How can we systematically deform?

16. Wednesday, October 17, 2012

16.1. **Ideals of points in the plane.**

Definition 16.1. The *Hilbert scheme* of n points in $\mathbb{C}^2$ is, as a set,

$$H_n = Hilb^n(\mathbb{C}^2) = \{\text{ideals } I \subset \mathbb{C}[x,y] : \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n.\}.$$

Actually H_n is a variety.

Example 16.2. Examples of points in H_n:

"most general": Take $\{P_1, \dots, P_n\} \subset \mathbb{C}^2$ and I its radical ideal.

"most special": Monomial Ideals $\leftrightarrow$ partitions of n.

Example 16.3. One such special ideal for $n = 4$ is as follows. Let $\lambda = 2+1+1$, a partition of 4. Its ideal is

$$I_\lambda = \langle x^2, xy, y^3 \rangle,$$

obtained by taking the generators of the complement of the Young diagram.

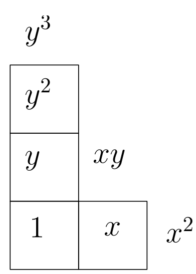


Figure 26. Young Diagram and corresponding Ideal Generators.

Set $V_m = \mathbb{C}\{\text{monomials of degree} \leq m\} = \mathbb{C}[x,y]_{\leq m} \cong \mathbb{C}^{\binom{m+2}{2}}$.

Lemma 16.4. *Given any colength n ideal I, the image of V_m span $\mathbb{C}[x,y]/I$ as a $\mathbb{C}$-vectorspace, provided $m \geq n$.*

Proof. The n monomials outside $in(I)$ span $\mathbb{C}[x,y]/I$ and these monomials must be in V_m. $\square$

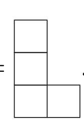
Remark 16.5. $I \cap V_m$ is a subspace of codimension n in V_m. It determines I.

H_n is a subset of the Grassmanian $Gr^n(V_m)$ of codimension n subspaces in V_m.

Definition 16.6. For a partition λ of n let $U_\lambda \subset H_n$ be the set of ideals such that

U_λ is defined by polynomials in an open cell of the Grassmannian.

Theorem 16.7. *The affine varieties U_λ form an open cover of the subset $H_n \subset Gr^n(V_m)$ for $M \geq n+1$, thereby endowing H_n with the structure of a (quasiprojective) variety.*

Example 16.8. Take $n = 4$ and $\lambda = 2+1+1 =$  .

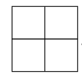
Every ideal $I \subset U_\lambda$ has the form

$$\langle x^2 - ay^2 - bx - py - q, xy - cy^2 - dy - ex - r, y^3 - fy^2 - gy - hx - s\rangle.$$

where $a, b, \dots, s$ are complex numbers that satisfy certain equations.

Namely, here the generators must form a Gröbner basis. Buchberger's S-pair criterion gives polynomial expressions for p, q, r, s in terms of $a, b, \dots, h$.

This implies that $U_\lambda = \text{Spec } \mathbb{C}[a, \dots, h] \cong \mathbb{C}^8$.

Example 16.9 (Difficult Exercise)**.** Let $\lambda = 2 + 2 =$ . U_λ is a smooth hypersurface in $\mathbb{C}^9$. The standard monomials are $xy, x, y, 1$.

One difference between this example and the last is that for $\lambda = 211$, there exists a weight vector under which the monomial ideal has higher weight than everything outside. No such order exists in this case.

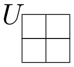
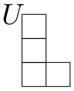
16.2. **Connectedness and Smoothness.**

Theorem 16.10. *The Hilbert scheme H_n is smooth and irreducible of dimension $2n$. (True only for points in $\mathbb{C}^2$)*

What about 120 points in $\mathbb{C}^3$? How about 8 points in $\mathbb{C}^4$? Both have trickier properties. The plane is also special in the applicability of partitions.

Lemma 16.11. *Every point $I \in H_n$ is connected to a monomial ideal by a rational curve.*

Proof. Gröbner bases. □

Example 16.12. For any $t \in \mathbb{C}$, consider $I_t = \langle x^2 - try, xy - t^2y^2, x^2y, xy^2, y^3\rangle$. This is in U for $t \neq 0$ but $I_0 = \langle x^2, xy, y^3\rangle \in U$.

Strategy for connecting two ideals $I, J \in H_n$:

(1) Send I and J to monomial ideals I_λ and I_μ respectively.
(2) Identify the sets of points corresponding to each monomial ideal.
(3) Homotope one set of points to the other.

We will see the explicit computation next class.

17. Friday, October 19, 2012

Proposition 17.1. *The Hilbert scheme H_n is connected.*

Proof. Every monomial ideal I_λ is the initial ideal of the radical ideal R_λ given by the points in λ.

$$R_\lambda = \langle x(x-1), xy, y(y-1)(y-2)\rangle.$$
$$I_\lambda = \langle x^2, xy, y^3\rangle = in(R_\lambda).$$

□

Proposition 17.2. *For each partition λ of n, the local ring has embedding dimension $\leq 2n$, i.e. the maximal ideal m_{I_λ} satisfies*

$$\dim_{\mathbb{C}}(m_\lambda / m_{I_\lambda}^2) \leq 2n.$$

Proof. The proof is a combinatorial game involving boxes and arrows. □

Together, these imply:

Theorem 17.3. *H_n is a smooth, irreducible variety of dimension $2n$.*

In 3- or higher-dimensional space, the upper bound of dimension of the Hilbert scheme will change, and it may be singular.

17.1. **Haiman's Theory.** Let the symmetric group Σ_n act on

$$\mathbb{C}[\mathbf{x},\mathbf{y}] = \mathbb{C}\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & & y_n \end{bmatrix}$$

The coordinate ring $S^n\mathbb{C}^2 = \mathbb{C}^{2n}/\Sigma_n$ is the invariant ring $\mathbb{C}[x,y]^{\Sigma_n} \subset \mathbb{C}[x,y]$.
An explicit list of generators of this ring is listed in Theorem 18.18.
For example:

$$\mathbb{C}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{\Sigma_2} = \mathbb{C}[x_1+x_2, x_1x_2, y_1+y_2, y_1y_2, x_1y_2+y_1x_2].$$

We obtain a morphism from:

$$\begin{tikzcd} & (\mathbb{C}^2)^n \arrow[d] \\ H_n \arrow[r] & S^n\mathbb{C}^2 \end{tikzcd}$$

This is an isomorphism outside the diagonal locus.

$$I_{\text{diag}} = \cap_{1\le i<j\le n}\langle x_i - x_j, y_i - y_j\rangle.$$

Theorem 17.4. *I_{diag} is generated by the polynomials*

$$\Delta_D(x,y) = \det\begin{bmatrix} x_1^{i_1}y_1^{j_1} & x_2^{i_1}y_2^{j_1} & \cdots & x_n^{i_1}y_n^{j_1} \\ x_1^{i_2}y_1^{j_2} & x_2^{i_2}y_2^{j_2} & \cdots & x_n^{i_2}y_n^{j_2} \\ \vdots & & & \vdots \\ x_1^{i_n}y_1^{j_n} & x_2^{i_n}y_2^{j_n} & \cdots & x_n^{i_n}y_n^{j_n} \end{bmatrix}$$

where $D = \{(i_1,j_1),\ldots,(i_n,j_n)\}$ runs over n-subsets of $\mathbb{N}^2$.

Open Problem: Find an elementary proof. Find explicit minimal generators.

Theorem 17.5 (Haiman's $n!$ and $(n+1)^{n-1}$ Theorem)**.** *(1) If $\lambda \vdash n$ then the vector space*

$$\mathbb{C}\left[\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_n},\right]\cdot\Delta_\lambda$$

has dimension $n!$.
(2) The vectorspace $\mathbb{C}[x,y]/\langle\mathbb{C}[x,y]_+^{\Sigma_n}\rangle$ has dimension $(n+1)^{n-1}$.

Key Player in the Proof. Isospectral Hilbert scheme X_n is the reduced fiber product

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (\mathbb{C}^2)^n \arrow[d] \\ H_n \arrow[r] & S^n\mathbb{C}^2 \end{tikzcd}$$

□

Exercise 17.6 (Exercise 18.3)**.** Compute the equations of the isospectral Hilbert scheme X_4 over $U_{\ydiagram{1,1,2}}$. (Similar to computation in Exercise 18.23.)

Theorem 17.7. *The isospectral Hilbert scheme is Gorenstein.*

$$\mathbb{C}\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]\cdot\Delta_\lambda \cong \mathbb{C}[x,y]/J_\lambda$$

is a 0-dimensional Gorenstein ring. Its scheme is the fiber of $X_n \to H_n$ over $I_\lambda \in H_n$. Therefore, this scheme has length $n!$.

18. Monday, October 22, 2012

18.1. **Haiman's Theory Continued.**

Corollary 18.1. *The Hilbert-Frobenius series of*

$$\mathbb{C}[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]\Delta_\lambda \cong \mathbb{C}[x,y]/J_\lambda$$

is the Macdonald polynomial $\tilde{H}_\lambda(z;q,t)$. Hence, $\tilde{H}_\lambda(z;q,t)$ is an $\mathbb{N}[q,t]$-linear combination of Schur functions $s_\mu(z)$.

Definition 18.2. The Hilbert-Frobenius series (in this context) is:

$$\sum_{i=1}^{\binom{n}{2}} \sum_{j=1}^{\binom{n}{2}} F_{ij}^\lambda(z) q^i t^j$$

where $F_{ij}^\lambda(z)$ is the sum (with multiplicity) of all Schur functions $s_\mu(z)$ for irreducible Σ_n-modules indexed by μ appearing in the degree (i,j) in $\mathbb{C}[x,y]/J_\lambda$.

Partitions index representations of the symmetric group. By duality, the same holds for the general linear group. If you set $z = q = t = 1$, then the Hilbert-Frobenius series evaluates to $n!$.

18.2. **Ehrhart Polynomials.**

Let P be a d-dimensional lattice polytope in $\mathbb{R}^d$ (i.e. the convex hull of a finite set of lattice points).

Theorem 18.3. *The function $E_P : \mathbb{N} \to \mathbb{N}, m \mapsto \#((m \cdot P) \cap \mathbb{Z}^d)$ is a polynomial of degree d, called the Ehrhart polynomial of P.*

Example 18.4. Consider the 3-cube $\mathrm{conv}(\{0,1\}^3)$. The Ehrhart polynomial is

$$E_{\text{cube}} = m^3 + 3m^2 + 3m + 1 = (m+1)^3.$$

Its inscribed octahedron (obtained by omitting antipodal vertices) has Ehrhart polynomial

$$E_{\text{oct}}(m) = \frac{2}{3}m^3 + 2m^2 + \frac{7}{3}m + 1.$$

The leading coefficient is the volume of the polytope, and the final coefficient is always 1.

Consider the semigroup $Q = \mathbb{N}\{(1,a) : a \in P \cap \mathbb{Z}^d\}$ in $\mathbb{Z}^{d+1}$. It need not be saturated, i.e. if Q is an "empty tetrahedron." (See Example 12.6).

Lemma 18.5. *E_p is the $\mathbb{N}$-graded Hilbert function of $K[Q_{sat}]$, i.e. for all*

$$m \in \mathbb{N} : E_p(m) = \dim_K(K[Q_{sat}]_m).$$

Proof. The monomials of degree m in $K[Q_{\text{sat}}]$ correspond to the lattice points in $m \cdot Q$. □

We need only show that the Hilbert function of $K[Q_{\text{sat}}]$ equals the Hilbert polynomials. By triangulation and inclusion-exclusion, this reduces to the case of simplices.

Lemma 18.6. *If P is a lattice simplex, then the $\mathbb{N}$-graded Hilbert function of $K[Q_{sat}]$ equals the Hilbert polynomials.*

Proof. Let $a_1, \ldots, a_{d+1} \in \mathbb{Z}^d$ be the vertices of P and L be the sublattice of $\mathbb{Z}^{d+1}$ spanned by $(1, a_1), \ldots, (1, a_n)$.

$S = [\mathbb{Z}^{d+1} : L]$ (the index of the lattice) is the volume of the half-open "cube"

$$B = \left\{ \sum_{i=1}^{d+1} \lambda_i (1, a_i) \mid 0 \leq \lambda_i < 1 \right\}.$$

$B \cap \mathbb{Z}^{d+1} = \{b_1, b_2, b_3\}$ is a set of representatives for the cosets of $\mathbb{Z}^{d+1}$ modulo L.

Set $x_i = t_0 \cdot t^{a_i} \Rightarrow K[Q_{\text{sat}}]$ is the free $K[x_1, \ldots, x_{d+1}]$ module with basis $\{t^{b_i} = t_0^{b_{i0}} \cdots t_d^{b_{id}} \mid i = 1, 2, \ldots, s\}$.

Note $\deg(t^{b_i}) = b_{i0} \leq d$ by definition fo B.

$$\Rightarrow E_p(m) = \sum_{j=1}^{s} \binom{(d - b_{j0}) + m}{d}.$$

Because $(d - b_{j0})$ is nonnegative, this is a polynomial in m. $\square$

18.3. **Dualizing Complexes.** Let $Q = Q_{\text{sat}}$. The canonical module of $K[Q]$ is the ideal

$$\omega_Q = \langle t^a : a \in \operatorname{int}(Q) \rangle.$$

(Related to canonical curves, canonical sheaves, etc.)

Theorem 18.7. *The following dualizing complex $\Omega_Q^\bullet$ is a minimal injective resolution of the canonical module:*

$$0 \to K\{\mathbb{Z}^d\} \to \bigoplus_{\textit{facets } F \subset Q} K\{T_F\} \to \bigoplus_{\textit{ridges } k \subset Q} K\{T_k\} \to \cdots \to \bigoplus_{\textit{rays } L \subset Q} K\{T_L\} \to K\{-Q\} \to 0,$$

where $T_G = G - Q$ is the injective hull of the face G of Q.

Question 18.8. What is the Hilbert series of the canonical module?

$$\dim_K(\omega_Q)_m = \#(\operatorname{int}(m \cdot P) \cap \mathbb{Z}^d) \text{ for } m > 0.$$

Example 18.9. Cube: $(m-1)^3 = m^3 - 3m^2 + 3m - 1$. (interior lattice points)

Octahedron: $\frac{2}{3}m^3 - 2m^2 + \frac{7}{3}m - 1$.

m	1	2	3	4	5	6	7
All Lattice Points	6	19	44	85	146	231	344
Interior Lattice Points	0	1	6	19	44	85	146

The Gorenstein property means that ω_C is principal.

19. Wednesday, October 24, 2012

19.1. **Brion's Formula.** Given a lattice polytope P in $\mathbb{R}^d$, its lattice point enumerator

$$\Phi_P(t) = \sum \{t^a : a \in P \cap \mathbb{Z}^d\}$$

Its interior lattice point enumerator is:

$$\Phi_P(t) = (-1)^{\dim(P)} \sum \{t^a : a \in \operatorname{int}(P) \cap \mathbb{Z}^d\}.$$

The inner tangent cone of P at a vertex v is

$$C_v = \mathbb{R}_{\geq 0}(P - v).$$

The vertex semigroup $Q_v = C_v \cap \mathbb{Z}^d$ is pointed with finite Hilbert basis $\mathcal{H}_v$ and corresponding polynomial ring

$$S_v := K[x_a | a \in \mathcal{H}_v] \twoheadrightarrow K[Q_v].$$

The S-module $K[Q_v]$ has K-polynomial $K_v(t)$ and universal denominator

$$D_v(t) = \prod_{a \in \mathcal{H}_v} (1 - t^a).$$

Hilbert series of $K[Q_v]$ is:

$$\frac{K_v(t)}{D_v(t)} = \sum_{a \in Q_v} t^a.$$

Theorem 19.1 (Brion)**.** *For all $m \in \mathbb{Z}$,*

$$\Phi_{mP}(t) = \sum_{v \in P \text{ vertex}} t^{m \cdot v} \frac{K_v(t)}{D_v(t)}$$

as a rational function in $\mathbb{Q}(t_1, \ldots, t_d)$.

Example 19.2. Let $d = 1, P = [2, 3]$.

$$\Rightarrow K_2(t) = 1, D_2(t) = 1 - t, K_3(t) = 1, D_3(t) = 1 - t^{-1}.$$

$$\Rightarrow \Phi_{mP}(t) = t^{2m} \frac{1}{1-t} + t^{3m} \frac{1}{1 - t^{-1}}.$$

For all $m \in \mathbb{Z}$, this is the correct Laurent polynomial.

For instance, taking $m = 7$, this is

$$\Phi_{7P}(t) = t^{14} \frac{1}{1-t} + t^{21} \frac{1}{1 - t^{-1}} = t^{14} \sum_{i=0}^{7} t^i.$$

Sketch of Proof.

$$Q \subset \mathbb{Z}^{d+1} \text{ generated by } \{(1, a) \mid a \in P \cap \mathbb{Z}^d\}$$

By Theorem 12.11, the Hilbert series of the canonical module is

$$H(\omega_Q; t_0, t) = \sum_{F \subset P \text{ faces}} (-1)^{d - \dim(F)} H(K\{T_F\}; t_0, t).$$

where $T_F = F - Q$.

The LHS equals $\sum t_0^{-m} (-1)^d \Phi_{mP}(t^{-1})$. The $F = \varnothing$ term on RHS equals $\sum_{m \geq 0} t_0^{-m} (-1)^{d+1} \Phi_{mP}(t^{-1})$.

Bring this to the LHS, multiply by $D = \prod_{v \in P \text{ vertex}} D_v(t^{-1})$ and apply Lemma 12.15. $\square$

Corollary 19.3 (Ehrhart Reciprocity)**.** *The number of interior lattice points in mP for $m \in \mathbb{N}$ equals $(-1)^{\dim P} E_P(-m)$ where E_P is the Ehrhart polynomial.*

Idea. Substittue $t_1 = \cdots = t_d = 1$ into Brion's formula, *very carefully* using L'Hopital's rule. $\square$

Software for calculating Brion's Formula, Ehrhart polynomials – Maple not so great, but Latte is good.

20. Friday, October 26, 2012

20.1. **Short Rational Generating Functions.**

Theorem 20.1 (Barvinok). *Suppose $d \in \mathbb{N}$ is fixed and $P \subset \mathbb{R}^d$ is a lattice polytope. The lattice point enumerator*

$$\Phi_P(t) = \sum_{v \in P \text{ vertices}} t^v \frac{K_v(t)}{D_v(t)}$$

can be computed in polynomial time in the binary bit complexity model.

Corollary 20.2. *In fixed dimension, the Ehrhart polynomial of a lattice polytope can be computed in polynomial time, specifically $O(m^{d/2})$.*

Example 20.3 (d= 2)**.** Let P be the convex hull of $\{(0,0),(0,a),(a,a^2)\}$, for $a \in \mathbb{N}$ (let a be very large).

$$\Phi_P(t_1,t_2) = \sum_{i=0}^{a} \sum_{j=0}^{i \cdot a} t_1^i t_2^j.$$

This is too big – exponential in $\log(a)$. We write equivalently:

$$= \frac{K_{00}}{(1-t_1)(1-t_1t_2^a)} + \frac{t_1^a}{(1-t_1^{-1})(1-t_2)} + \frac{t_1^a t_2^a}{(1-t_1^{-1}t_2^a)(1-t_2^{-1})}.$$

$$K_{00} = t_1t_2 + t_1t_2^2 + \cdots + t_1t_2^{a-1}.$$

This is still exponential size, but we can rewrite it as a fraction:

$$= 1 + \frac{t_1t_2 - t_1t_2^a}{1-t_2}.$$

Idea of Proof. Triangulate P in polynomial time (d fixed). When P is a simplex, need only compute $d+1$ series like K_v.

Lemma 20.4. *If $a_1, \ldots, a_d$ are linearly independent, then the Hilbert series of $Q = \mathbb{R}_{\geq 0}\{a_1, a_2, \ldots, a_d\} \cap \mathbb{Z}^d$ is a short rational generating function that can be computed in polynomial time.*

Idea: Write the cone Q as an alternating sum of a few unimodular cones, as demonstrated in 27.

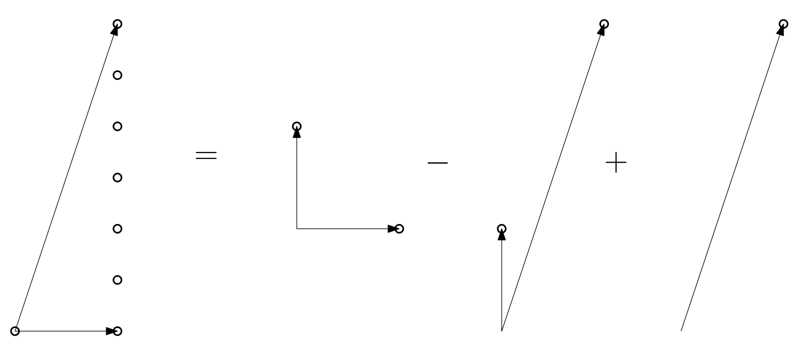


Figure 27. Decomoposition into unimodular cones.

□

Theorem 20.5 (Barvinok-Woods). *For d fixed, the Hilbert basis $\mathcal{H}_Q$ of any saturated affine semigroup $Q \subset \mathbb{Z}^d$ can be computed in polynomial time. (We have a "short encoding".)*

Theorem 20.6. *Fix d and n. Let $A \in \mathbb{Z}^{d\times n}$ and $L = \ker(A)$. The following can be computed in polynomial time:*

(1) the $\mathbb{Z}^d$-graded Hilbert series of S/I_L. (This would allow you to compute the Frobenius number – the largest number not in the lattice.)
(2) Any reduced Gröbner basis of I_L.
(3) A finite universal Gröbeer basis.

Example 20.7 (n=4,d=2)**.** Given $a \in \mathbb{N}$,

$$G(x,y) = x_1x_4y_2y_3x_4^{a-2}y_3^{a-1} + \frac{x_1x_3y_2^2((x_1y_2)^{a-2} - (x_3y_4)^{a-2}}{x_1y_2 - x_3y_4}.$$

Given by the matrix:

$$A = \begin{bmatrix} a & a-1 & 1 & 0 \\ 0 & 1 & a-1 & a \end{bmatrix}. \qquad \{(s^a : s^{a-1}t : st^{a-1} : t^a)\} \subset \mathbb{P}^3.$$

21. Monday, October 29, 2012

21.1. **The Complete Flag Variety.** $\mathcal{F}l_n =$ the set of complete flags of linear subspaces.

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = K^n.$$

We will realize $\mathcal{F}l_n$ as a subvariety of a product of projective spaces:

$$\mathcal{F}l_n \subset \mathbb{P}^{n-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{3}-1} \times \cdot \times \mathbb{P}^{\binom{n}{n-1}-1}.$$

For $\sigma \subset [n]$ and $\Theta \in K^{n\times n}$ let Θ_σ be the submatrix with rows $1,2,\ldots,d$ and columns σ. Their determinants are the Plücker coordinates of Θ, and they define a point in the product of projective spaces. This point depends only on the flag $V_\bullet$, where V_i is the span of the first d rows of Θ. The projection of $\mathcal{F}l_n$ into $\mathbb{P}^{\binom{n}{d}-1}$ is the Grassmannian of d-planes in K^n.

Question 21.1. What is the dimension of $\mathcal{F}l_n$?

Answer: Consider each subspace as a hyperplane with normal vector unique up to scaling. Then it is clear that we have $(n-1) + (n-2) + \cdots + 1 = \binom{n}{2}$.

Definition 21.2. Let $\mathbf{x}$ be an $n \times n$ matrix of variables. The subalgebra of $K[\mathbf{x}]$ spanned by the 2^n Plücker coordinates of $\mathbf{x}$ is the Plücker algebra.

21.2. **Quadratic Plücker Relations.** The Plucker ideal I_n is the Kernel of the map

$$K[p] \to K[\mathbf{x}], \qquad p_\sigma \mapsto \det x_\sigma.$$

Example 21.3. $n = 3$.

$$I_3 = \langle p_{23}p_1 - p_{13}p_2p_{12}p_3$$

The proof is that duplicating a row will give a 3×3 matrix with zero determinant. This is another expression for that determinant.

Example 21.4. $n = 4$. Our variety $\mathcal{F}l_n \subset \mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. I_4 is minimally generated by 10 quadrics, including

$$p_{234}p_{13} - p_{134}p_{23} + p_{123}p_{34}.$$

This can be alternatively written as:

$$\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&3\\\cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&3\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&4\\\cline{1-2}\end{array}.$$

Definition 21.5. *Young's Poset* $\mathcal{P}$ on $2^{[n]}$ is defined by

$$\sigma = \{\sigma_1 \leq \cdots \leq \sigma_s\} \leq \tau = \{\tau_1 \leq \cdots \leq \tau_t\}$$

if $s \geq t$ and $\sigma_i \leq \tau_i$ for $i = 1, \ldots, t$.
The chains in $\mathcal{P}$ are semistandard tableaux.

The Hasse diagram of the Young Poset for I_4 is in Figure 28

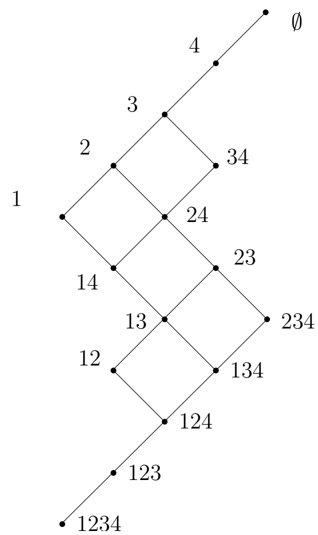


FIGURE 28. Hasse Diagram for I_4.

Theorem 21.6. *The ideal I_n of Plücker relations has a quadratic Gröbner basis (under lex term order), where the initial ideal in(I_n) is generated by products $p_\sigma p_\tau$ of incomparable pairs.*

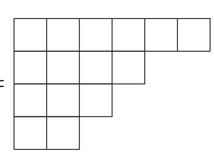
Example 21.7. In I_4 if we consider the simplicial complex of the Hasse diagram, the chains are the facets. So the number of facets = the number of chains (12) is the degree of the ideal.

Outline of Proof. (1) For every incomparable pair $\{\sigma, \tau\}$ of $\mathcal{P}$, there exists $f \in I_n$ such that $\text{in}(f) = p_\sigma p_\tau$.
(2) Fix the purely lex term order on $K[\mathbf{x}]$ with

$$x_{11} > x_{12} > \ldots, x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{n1} > \cdots > x_{nn}.$$

Then $\text{in}(\phi_n(p_\sigma)) = x_{1\sigma_1} x_{2\sigma_2} \cdots x_{s\sigma_s}$.
(3) For any monomial p^a in $K[p]$ we have

$$\text{in}(\phi_n(p^a)) =$$



i.e. some set of variables in the upper left corner.
(4) There is a unique semistandard monomial p^b such that $\text{in}(\phi_n(p^b))$. Hence, these p^b's are linearly independent modulo I_n

□

Corollary 21.8. *in(I_n) is the Stanley-Reisner ideal of the order complex of $\mathcal{P}$.*

Corollary 21.9. *The semistandard monomials p^a form a K-basis for the Plücker algebra.*

22. Wednesday, October 31, 2012

(We are discussing the fact that the minors form a Sagbi basis. Some material has been missed.)

23. Friday, November 2, 2012

23.1. **Gelfand-Tseitlin Semigroups.**

Definition 23.1. A *GT Pattern* is an array $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$\lambda_{ij} \geq \lambda_{i,j+1} \geq \lambda_{i+1,j} \geq 0$$

for all i,j and $\lambda_{ij} = 0$ if $i+j > n+1$.

GT_n = the semigroup of integer solutions. The dimension is $\binom{n+1}{2}$.

Proposition 23.2. *The Gelfand-Tseitlin semigroup GT_n has Hilbert basis $\mathcal{H}_n^1$ consisting of partitions with distinct parts fitting into an $n \times n$ array.*

Theorem 23.3. *There is an automorphism of $\mathbb{Z}^{n\times n}$ that takes GT_n to the antidiagonal semigroup corresponding tot he initial algebra $in(R)$ of the Plücker algebra R.*

Corollary 23.4. *$in(R)$ is isomorphic to the GT semigroup algebra $K[GT_n]$.*

Corollary 23.5. *Both $in(R)$ and R are Cohen-Macaulay.*

Example 23.6.

$$\mathcal{H}_3^1 = \left\{ \ydiagram{1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{2,1}, \ydiagram{3,1}, \ydiagram{3,2}, \ydiagram{3,2,1} \right\}.$$

$$\mathcal{H}_3^1 = \left\{ \begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ 1 & & \\ & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \right\}.$$

These in turn correspond to $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}, p_{123}$.
$in(R) = K[p]/\langle p_{12}p_3 - p_{13}p_2\rangle J_3$ of dimension 6.

23.2. **Back to Section 14.3.**

Monomial map $\psi_n : K[p] \to R \subset K[x]$, sending $p_\sigma \mapsto in(\det(x_\sigma))$.
The toric ideal $J_n = \ker(\psi_n)$.
Our quadratic Gröbeer Basis for the Plücker ideal I_n in Theorem 14.6 factors through a Gröbeer Basis for J_n consisting of quadratic binomials.

Remark 23.7. Regarding the Combinatorics –
Young's poset is a *distributive lattice* $(P, \wedge, \vee)$. If $\sigma = \{\sigma_1 < \cdots < \sigma_s\}$ and $\tau = \{\tau_1 < \cdots < \tau_t\}$ with $s \geq t$ then

$$\sigma \wedge \tau = \{\min\{\sigma_i, \tau_i\}, i = 1, \ldots, t\} \cup \{\sigma_{t+1}, \ldots, \sigma_s\}.$$

$$\sigma \vee \tau = \{\max\{\sigma_i, \tau_i\}, i = 1, \ldots, t\}.$$

We define a partial term order on $K[p]$ via $p^a \leq p^b$ iff $in(\phi_n(p^a)) \leq_{diag} in(\phi_n(p^b))$.

Theorem 23.8. *We have $J_n = in(I_n)$, the reduced Gröbeer basis of J_n under revlex(P) consists of all binomials $p_\sigma p_\tau - p_{\sigma\wedge\tau}p_{\sigma\vee\tau}$ where τ, σ are incomparable in Young's poset.*

Remark 23.9. Standard monomial theory leads to applications in Representation Theory. This last theorem represents the confluence of Gröbner basis theory with Standard monomial theory.

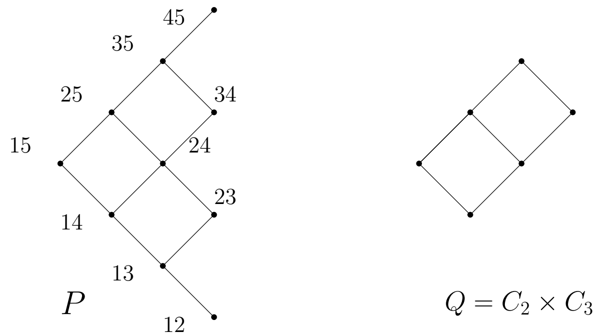


FIGURE 29. Hasse diagram for the Grassmannian.

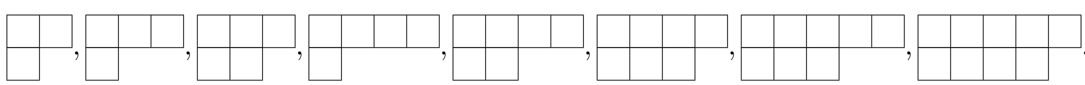
Example 23.10 (Prof. Sturmfels' Favorite Example)**.** Consider the Grassmannian $Gr(2,5)$. By the fundamental theorem of distributive lattices, every distributive lattice comes from the order ideal of some other poset.

$P = J(Q)$, where $Q = C_2 \times C_3$.

We can take monomials to binomials to trinomials. The five incomparable pairs are in the left column, and we grow these to Plucker relations:

$$\begin{array}{lll} p_{14}p_{23} & -p_{13}p_{24} & +p_{12}p_{34} \\ p_{15}p_{23} & -p_{13}p_{25} & +p_{12}p_{35} \\ p_{15}p_{24} & -p_{14}p_{25} & +p_{12}p_{45} \\ p_{15}p_{34} & -p_{14}p_{35} & +p_{13}p_{45} \\ p_{25}p_{34} & -p_{24}p_{35} & +p_{23}p_{45} \end{array}$$

There is a bijection between order ideals in Q and partitions with two distinct parts:



24. Monday, November 5, 2012 – Guest Lecture by Adam Boocher

Definition 24.1. Determinantal ideals are ideals generated by sub-determinants of a generic matrix.

Notation 24.2.

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1l} \\ \vdots & \ddots & \\ x_{k1} & \cdots & x_{kl} \end{pmatrix}.$$

$$M_{kl} = \{k \times l \text{ matrices}/\mathbb{C}\}.$$

Definition 24.3. Let X_r denote the set of matrices $M \in M_{kl}$ such that $\operatorname{rank}(M) \le r$.

Fact: This is cut out by the $(r+1)\times(r+1)$ minors of X.

$$\text{rank } M \le r \quad \Leftrightarrow \quad (r+1)\times(r+1) \text{ minors of } M \text{ vanish.}$$

Therefore, $I(X_r) = \sqrt{(r+1)\text{-minors of } X}$.

Example 24.4. Let $k = l = 3$. Then $X_3 = M_{33} \cong \mathbb{C}^9$.

$X_2 = V(\det X) = 8$-dimensional variety.

$X_1 = V(\Delta_1, \ldots, \Delta_9) = 5$-dimensional variety.

$X_0 - \{0\}$.

Proposition 24.5.
$$\dim X_r = r(k+l-r).$$

Proof. Let the first r rows of the matrix be linearly independent; then the next $k-r$ rows must be linear combinations of the first r. This gives us rl degrees of freedom in choosing the first r rows, and $r(k-r)$ degrees of freedom in choosing the multipliers in the final rows. □

Corollary 24.6. *codim*$(X_r) = (k-r)(l-r)$.

Proof. Just subtract the dimension from kl. □

Theorem 24.7. *X_r is an irreducible variety, and $I(X_r)$ is the ideal of $(r+1)\times(r+1)$ minors of X.*

Proof. The proof proceeds in two steps:

(1) $I(X_r) =$ a prime ideal.
(2) $\sqrt{(r+1)\text{-minors of } X} = I(X_r) = ((r+1)\text{-minors of } X)$.

Let $M \in X_r$, this means that rank $M \le r$.

$$\begin{tikzcd} \mathbb{C}^l \arrow[rr, "M"] \arrow[dr, "B"'] & & \mathbb{C}^k \\ & \mathbb{C}^r \arrow[ur, "A"'] & \end{tikzcd}$$

where $M = AB$.

$$\mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^r) \times \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) \twoheadrightarrow X_r$$

is surjective.

A surjective image of an irreducible variety is irreducible. To see what is happening in rings, we are factoring a matrix into:

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix} = (x_{ij})$$

Then the map of polynomial rings sends $x_{11} \mapsto ag + dj, \ldots$

How do we show that $J =$ (minors) is radical? It suffices to show that an initial ideal of J is radical. Find a Gröbner basis for J. □

24.1. **Matrix Schubert Varieties.** Motivational note: Why should we differentiate between diagonal and antidiagonal term orders in a generic matrix? This is because in the generalization to Schubert varieties, we give preference to one corner, so the difference is important.

Definition 24.8. A partial permutation matrix w is a matrix of 0s and 1s with at most one 1 in each row and column.

Definition 24.9.
$$\bar{X}_w = \{Z \in M_{kl} |\ \mathrm{rank}(Z_{p\times q}) \le\ \mathrm{rank}(w_{p\times q})\}$$
where Z_{pq} is the upper left $p \times q$ submatrix.

Example 24.10. Let $w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We define the matrix $r(w)$ by letting the (i,j)-th entry be the rank of the upper left submatrix with bottom right corner (i,j). Here,

$$r(w) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

In this matrix, it is obvious that the first row and column must be filled with 1's by the rook-placement condition. Similarly, the bottom right corner must be 2, since that contains the total number of 1s. The only non-obvious entry is the 1 in position $(2, 2)$. This implies that

$$\bar{X}_w = V(x_{11}x_{22} - x_{12}x_{21}).$$

Example 24.11. $X_r = X_w$ when

$$w = \begin{pmatrix} 1 & & & \\ & {\scriptstyle r} & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix}$$

Example 24.12. Consider the set M_{33}, i.e. $k = l = 3$. There are six permutation matrices (S_3). Each one indicates a different ideal. The most special one is:

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Unlike the other permutations which require specified entries to be 0, this one requires the upper 2×2 minor to be 0.

Proposition 24.13. *Every partial permutation matrix w can be extended to a full square "honest" permutation matrix w' such that*

$$mingens(I_w) = mingens(I_{w'}).$$

25. Wednesday, November 7, 2012 / Guest Lecture by Pablo Solis

Last time: we defined Matrix Schubert varieties $\overline{X_w}$ and Schubert Ideals $I_w = I(\overline{X_w})$.

Today, we will discuss multidegree of $\overline{X_w}$, and $C(\overline{X_w}, t, s)$.

Theorem 25.1. *(Theorem/Defintion)*

$$\mathcal{S}_w = C(\overline{X_w}, t, s).$$

We will

- review $\overline{X_w}, I_w$.
- Explain the multidegree $\overline{X_{w_0}}$.
- Bruhat order on S_n.
- Double Schubert polynomials.

25.1. Schubert Varieties and Ideals.

Let $w \in \mathbb{C}^{k \times l}$, for example $w_1 = [\ 0 \quad 0\ ], w_2 = [\ 0 \quad 1\ ]$.

The variety $\overline{X_w}$ is the set

$$\left\{ A \in \mathbb{C}^{k\times l} \mid rk(A_{pq}) \leq rk(w_{pq}) \right\}.$$

where A_{pq} and w_{pq} are given by the upper-left submatrices

$$\begin{bmatrix} & | & & \\ - & A_{pq} & & \\ & & & \end{bmatrix}, \begin{bmatrix} & | & & \\ - & w_{pq} & & \\ & & & \end{bmatrix}$$

For our cases above: $\overline{X_{w_1}} = [\ 0 \quad 0\ ], \overline{X_{w_2}} = [\ 0 \quad a\ ]$, for any $a \in \mathbb{C}$.

$$w_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R(w_3) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

In this case,

$$\overline{X_{w_3}} = \left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \quad \middle| \quad ad - bc = 0 \right\}.$$

$$\Rightarrow I_{w_3} = (x_{11}, x_{12}, x_{21}, x_{22}, x_{13}x_{24} - x_{14}x_{23}).$$

25.2. **Multidegree.** Let $w_0 = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in S_n$.

Then, Adam proved last time that $\overline{X_{w_0}}$ has $I_{w_0} = (x_{ij} \mid i + j \leq n)$.

Grading on $k[x_{ij}]$. Let

$$\mathbb{Z}^{l+k} = \bigoplus_{i=1}^{k} t_i\mathbb{Z} \bigoplus_{j=1}^{l} s_j\mathbb{Z}.$$

Then the $\deg(x_{ij}) = t_i - s_j$.

Remark 25.2 (Important point)**.** It is enough to consider $w \in S_n$, since any partial permutation w extends to $\tilde{w} \in S_n$ such that $\text{mingens}(I_w) = \text{mingens}(I_{\tilde{w}})$.

Theorem 25.3.

$$C(S/(x_{i_1}, \ldots, x_{i_m}), t) = \deg(x_{i_1}) \cdots \deg(x_{i_m}).$$

$$\Rightarrow C(\overline{X_{w_0}}, t, s) = \prod_{i+j \leq n} t_i - s_j.$$

Example 25.4. Take $w_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then, $C(\overline{X_{w_0}}, t, s) = (t_1 - s_1)(t_1 - s_2)(t_2 - s_1)$.

The general idea going forward is that any permutation w can be factored as a sequence of transpositions. These transpositions can then be considered as operators acting on the ts in the multidegree of our w_0. Let $f = f(t_1, \ldots, t_k, s_1, \ldots, s_l)$. Then,

$$\partial_i f = \frac{f - f(t_1, \ldots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \ldots)}{t_i - t_{i+1}}.$$

$$\mathcal{S}_w = \partial_{i_1} \cdots \partial_{i_m} \prod_{i+j \leq n} (t_i - s_j).$$

This is called the double Schubert polynomial.

Theorem 25.5 (Definition/Theorem)**.** *$\mathcal{S}_w$ is well-defined and equal to $C(\overline{X_w}, t, s)$.*

25.3. **Bruhat Order.** Let $\sigma_i = (\ i \quad i+1\ )$ be a generator of S_n.

Any permutation can factor as a sequence of transposition in a minimal way, this is called a reduced expression.

Example 25.6. Let $w_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then, $\sigma_1 w_0$.

$$C(\overline{X_{\sigma_1 w_0}}, t, s) = \partial_1 (t_1 - s_1)(t_1 - s_2)(t_2 - s_1) = (t_1 - s_1)(t_2 - s_2).$$

The Bruhat order helps us figure out a reduced expression for the permutation.

If you take a permutation with all zeros, except for an $r \times r$ upper-left identity matrix, then the $\mathcal{S}_w$ = the Schur polynomial.